MATH 185 Lecture Notes

Alec Li

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Lecture 1

Complex Numbers

Today we'll go through an introduction of complex analysis. The difference between this class (complex analysis) and MATH 104 (real analysis) is that we're talking about complex numbers; specifically, functions $f: \mathbb{C} \to \mathbb{C}$. This change gives us a lot of different properties of functions.

1.1 Holomorphic Functions

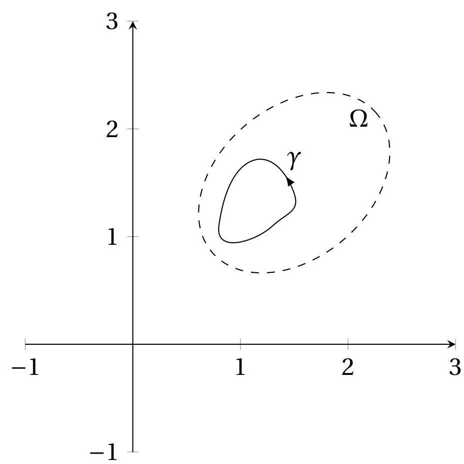
Later, we're going to spend a lot of time talking about *holomorphic functions*. If we consider the real analog $f: \mathbb{C} \to \mathbb{C}$, we can classify f to be *differentiable* by looking at the derivative; a similar classification can be made with complex functions, looking at differentiability in the complex sense—we call these functions holomorphic functions. As such, holomorphic functions are the complex analog to differentiable functions in real analysis.

These functions may seem similar at first, but their properties are actually quite different, and holomorphic functions behave differently.

1.1.1 Contour Integration

Most of the functions we'll be talking about are of the form $f: \Omega \to \mathbb{C}$ where Ω is an open set in $\mathbb{C}$ (defined similarly to an open set in $\mathbb{R}^2$).

Suppose this function is holomorphic, and let us consider a closed loop γ in the open set Ω.



We always have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

This is called Cauchy's Theorem for holomorphic functions.

In the real case, instead of taking a holomorphic function f, we can take a pair of real functions (corresponding to the real and imaginary parts of the function) $p(x, y)$ and $q(x, y)$. We can take the same path and take the integral along the loop:

$$\int_\gamma P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = -\iint_D \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,\mathrm{d}x\,\mathrm{d}y.$$

The RHS comes from Green's formula, which is not always zero (otherwise multivariable calculus would be quite boring).

Example 1.1

If we have $P(x,y) = -y$ and $Q(x,y) = x$, we have $-\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 2$, which means that the integral evaluates to $2 \cdot \text{Area}(D) \neq 0$.

1.1.2 Regularity

In real analysis, we've considered functions that are differentiable but not second-differentiable. However, if we have a holomorphic function f, we know that f is always *infinitely* differentiable.

Example 1.2

Suppose we have the real function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

We can see that f is first-differentiable:

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

However, f' is not continuous, and thus is not differentiable.

In the complex case, this will never happen; f', f'', f''', etc. will *always* exist.

1.1.3 Analytic Continuation

Let us again take the complex plane and take an open set Ω in the plane. Suppose we have holomorphic f, g defined in the neighborhood of Ω, i.e. defined on $\tilde{\Omega} \supseteq \Omega$. If $f = g$ on Ω, then $f = g$ on $\tilde{\Omega}$ as well.

This is called analytic continuation; there is only one way to extend the function to $\tilde{\Omega}$ from Ω.

However, this is not the case for functions on $\mathbb{R}^2$; if Ω is the unit disk, then we have the two functions

$$f(x,y) = 0$$
$$g(x,y) = \begin{cases} 0 & \text{on unit disk} \\ e^{\frac{1}{1-x^2-y^2}} & \text{outside of unit disk} \end{cases}$$

We can show that g is infinitely differentiable, and $f = g$ on Ω, but $f \neq g$ outside of Ω.

1.2 Complex numbers

In this class, we'll mostly be talking about *holomorphic* functions defined on *open sets* in the *complex plane*. The three italicized terms will be what we're focusing on first, starting with the complex numbers.

Complex numbers have a bijection between pairs of real numbers; that is, $\mathbb{C} \iff \mathbb{R}^2$ have a one-to-one correspondence.

The set of complex numbers is defined as $\{x + iy \mid x, y \in \mathbb{R}\}$, and the set of pairs of real numbers is defined as $\{(x,y) \mid x, y \in \mathbb{R}\}$. As such, (x,y) and $x + iy$ are equivalent representations.

Complex numbers are usually denoted by $z = x + iy$ for some $x, y \in \mathbb{R}$. Here, x is the real part of z, and y is the imaginary part of z; equivalently notated as $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

We also have an injection from $\mathbb{R} \hookrightarrow \mathbb{C}$ with $x \mapsto x + i \cdot 0$. As such, we can consider $\mathbb{C}$ as an *extension* from $\mathbb{R}$. (For example, in $\mathbb{R}$, not all polynomials can be factored, but in $\mathbb{C}$ all polynomials can be factored into degree one terms.)

1.2.1 Operations on $\mathbb{C}$

On the reals, we have operations $+$, $-$, $\cdot$, and $/$. On the complexes, we can define these operations similarly:

Definition 1.3: Addition and Subtraction on $\mathbb{C}$

We define addition and subtraction in a similar manner to the reals.

Addition: $z_1 + z_2 = (\mathrm{Re}(z_1) + \mathrm{Re}(z_2)) + (\mathrm{Im}(z_1) + \mathrm{Im}(z_2))i$

Subtraction: $z_1 - z_2 = (\mathrm{Re}(z_1) - \mathrm{Re}(z_2)) + (\mathrm{Im}(z_1) - \mathrm{Im}(z_2))i$

Definition 1.4: Multiplication on $\mathbb{C}$

Trying to define multiplication on $\mathbb{R}^2$, the most naive way is to define $(a_1, b_1)\cdot(a_2, b_2) = (a_1a_2, b_1b_2)$. However, this multiplication is not very good; we can get a zero pair from two nonzero pairs: $(1,0)\cdot(0,1) = (0,0)$.

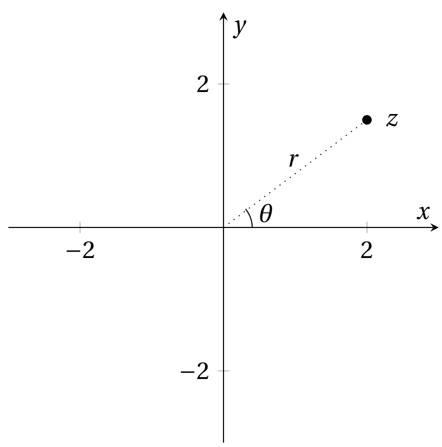
As such, we define multiplication on complex numbers in a different way; we distribute the terms:

$$\begin{aligned}(a_1 + b_1 i)\cdot(a_2 + b_2 i) &= a_1\cdot(a_2 + b_2 i) + b_1 i(a_2 + b_2 i)\\ &= a_1a_2 + (a_1)(b_2 i) + (b_1 i)(a_2) + (b_1 i)(b_2 i)\\ &= a_1a_2 + (a_1b_2)i + (a_2b_1)i + (b_1b_2)(i\cdot i) \qquad \text{(associativity, commutativity)}\end{aligned}$$

At this point, we define $i \cdot i = -1$ in the complex numbers, and keep going:

$$\begin{aligned} &= a_1a_2 + a_1b_2 i + a_2b_1 i - b_1b_2\\ (a_1 + b_1 i)\cdot(a_2 + b_2 i) &= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i\end{aligned}$$

A more intuitive meaning to the definition $i \cdot i = -1$ can be done through polar coordinates:



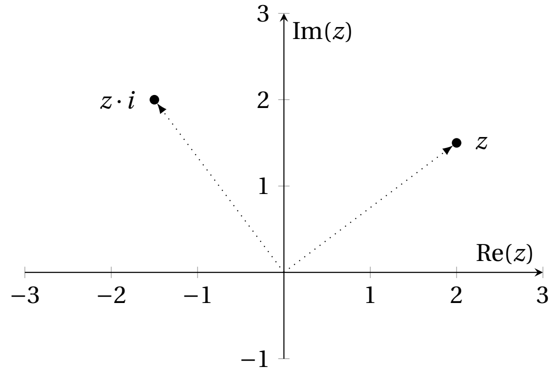
Here, we define $r = |z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$ and $\theta = \arg(z)$.

By Euler's formula, we have

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta)\\ re^{i\theta} &= r(\cos\theta + i\sin\theta)\\ &= r\cos\theta + ir\sin\theta\\ &= \mathrm{Re}(z) + i\,\mathrm{Im}(z)\end{aligned}$$

We also have that $z = i = 1\cdot e^{i\cdot\frac{\pi}{2}}$, and $-1 = e^{i\pi}$.

Geometrically, all we're doing is rotating counter-clockwise by $\frac{\pi}{2}$ (90 degrees):



Generally, we have $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

Before going into division, we must define the conjugate of a complex number:

Definition 1.5: Conjugate

The *conjugate* $\bar{z}$ of the complex number z is defined as $\bar{z} = \operatorname{Re}(z) - i\operatorname{Im}(z)$.

Definition 1.6: Division on $\mathbb{C}$

As with division in the reals, to calculate z_1/z_2, we must have $z_2 \neq 0$.

Let us now consider $z \cdot \bar{z}$; we can see that

$$z \cdot \bar{z} = (\operatorname{Re}(z) + \operatorname{Im}(z))(\operatorname{Re}(z) - \operatorname{Im}(z)) = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 = |z|^2 \in \mathbb{R}.$$

As such, we can multiply the numerator and denominator by $\overline{z_2}$ to simplify the calculation, giving us

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}.$$

1.2.2 More on Conjugates

Notice

$$z + \bar{z} = \operatorname{Re}(z) + i\operatorname{Im}(z) + \operatorname{Re}(z) - i\operatorname{Im}(z) = 2\operatorname{Re}(z).$$

This means that we have $\operatorname{Re}(z) = \frac{1}{2}\left(z + \bar{z}\right)$.

Similarly, we have

$$z - \bar{z} = \operatorname{Re}(z) + i\operatorname{Im}(z) - \operatorname{Re}(z) + i\operatorname{Im}(z) = 2i\operatorname{Im}(z).$$

This means that we have $\operatorname{Im}(z) = \frac{1}{2i}\left(z - \bar{z}\right)$.

1.2.3 More on Norms

We've defined $\bar{z}$, the norm of z, earlier. This quantity geometrically represents the distance from z to the origin (0).

In a similar notion, we can define the distance between $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ as $|z_1 - z_2| = d(z_1, z_2)$. This evaluates to

$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

In 104, we talked about the concept of a metric space. On $\mathbb{R}$, the distance function is the absolute value function. On $\mathbb{R}^2$, the distance function is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

The distance function on $\mathbb{C}$ is thus defined similarly to $\mathbb{R}^2$, as $d(z_1, z_2) = |z_1 - z_2|$. This makes $\mathbb{C}$ a metric space.

Briefly, a metric space is defined on a set S with a distance function $d : S \times S \to \mathbb{R}$ such that (for $x, y, z \in S$):

- $d(x, y) \ge 0$, with $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- Triangle inequality: $d(x, y) + d(y, z) \ge d(x, z)$

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Lecture 2

The Metric Space $\mathbb{C}$

2.1 Convergence

Definition 2.1: Convergence in $\mathbb{C}$

Let us consider a sequence $\{z_n\}$ of complex numbers. We say $\{z_n\}$ is convergent in $\mathbb{C}$ if we can find some $w \in \mathbb{C}$ such that $\{d(z_n - w)\} = \{|z_n - w|\}$, a sequence of real numbers, converges to 0.

We write this as $\lim_{n\to\infty} z_n = w$, where w is called the limit of $\{z_n\}$.

Note that if $\{z_n\}$ is convergent, then the limit is unique.

Since we can write any complex number as $z_n = \text{Re}(z_n) + i\,\text{Im}(z_n)$, any sequence $\{z_n\}$ in $\mathbb{C}$ is equivalent to two sequences in $\mathbb{R}$: $\{\text{Re}(z_n)\}$ and $\{\text{Im}(z_n)\}$.

As such, if $\{z_n\}$ converges to w, then we must have $\{\text{Re}(z_n)\} \to \text{Re}(w)$ and $\{\text{Im}(z_n)\} \to \text{Re}(w)$ (the converse holds as well).

2.2 Cauchy Sequences

Definition 2.2: Cauchy Sequence

$\{z_n\}$ is a Cauchy sequence if for any $\varepsilon > 0$, we can find some $N \in \mathbb{Z}^+$ such that any two $m, n \in \mathbb{Z}^+$ for $m, n > N$ satisfies

$$|z_m - z_n| < \varepsilon.$$

Theorem 2.3: Convergent Sequences are Cauchy Sequences

If we have a sequence $\{z_n\}$ that converges to $w \in \mathbb{C}$, then $\{z_n\}$ must be a Cauchy sequence.

Proof. Since $z_n \to w$, for any $\varepsilon > 0$, there exists some N such that any $m > N$ satisfies $|z_m - w| < \frac{\varepsilon}{2}$. The same applies for any $n > N$, where $|z_n - w| < \frac{\varepsilon}{2}$.

By the triangle inequality, we have

$$|z_m - z_n| = |(z_m - w) - (z_n - w)| \le |z_m - w| + |z_n - w| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As such, for any $\varepsilon > 0$, we've just found an N such that any $m, n > N$ satisfies $|z_m - z_n| < \varepsilon$, and $\{z_n\}$ is a Cauchy sequence.

Note that we only used the fact that $|\cdot|$ is a distance in a metric space (to apply the triangle inequality); as such, this holds for any metric space with any distance function. □

A followup question: is every Cauchy sequence convergent? No, not for any arbitrary metric space. However, it is true for $\mathbb{C}$ with the usual distance function.

This means that $(\mathbb{C}, d)$ is *complete*.

Definition 2.4: Complete

A metric space (S, d) is *complete* if every Cauchy Sequence converges in the metric space.

Theorem 2.5: $\mathbb{C}$ is complete

$(\mathbb{C}, d)$ is complete.

Proof. Suppose we have a Cauchy sequence $\{z_n\}$. Since we know $\{z_n\}$ is equivalent to $\{\text{Re}(z_n)\}$ and $\{\text{Im}(z_n)\}$, one can see that $\{\text{Re}(z_n)\}$ and $\{\text{Im}(z_n)\}$ are also Cauchy sequences.

Since $\mathbb{R}$ is complete, then both $\{\text{Re}(z_n)\} \to a$ and $\{\text{Im}(z_n)\} \to b$ are convergent. As such, $\{z_n\}$ must also converge to some $a + ib \in \mathbb{C}$.

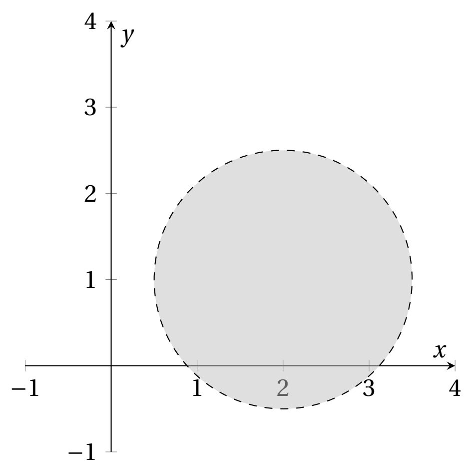
As such, the completeness of $\mathbb{C}$ is a direct corollary of the completeness of $\mathbb{R}$. $\square$

2.3 Open sets in $\mathbb{C}$

2.3.1 Notation

An object we talk about often in $\mathbb{C}$ is a disk. A disk is defined by a center z_0 with a radius $r > 0$. Notationally, we call this set

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$



A special case is when $r = 0$, meaning $D_0(z_0) = \varnothing$.

Another set we consider is a "closed" disk of radius r centered at z_0, defined as

$$\overline{D_r(z_0)} := \{z \in \mathbb{C} \mid |z - z_0| \le r\}.$$

Again, a special case of $r = 0$ gives us $\overline{D_r(z_0)} = \{z_0\}$.

Another bit of notation is for the *unit disk*, or the disk centered at $z_0 = 0$ with radius $r = 1$; we define this as $D_1(0) = \mathbb{D}$.

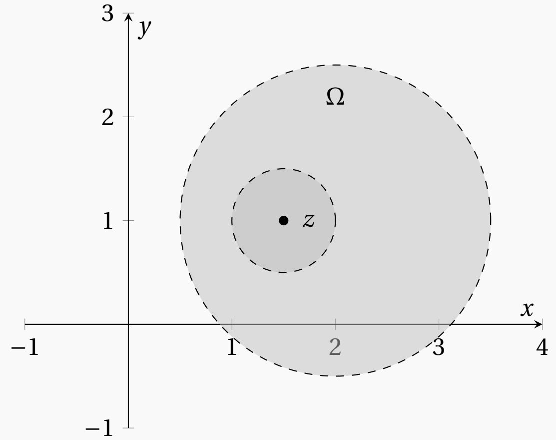
2.3.2 Interior points

Definition 2.6: Interior Point

Let us consider a set $\Omega \subseteq \mathbb{C}$. A point $z \in \omega$ is called an *interior point* if there is some $r > 0$ such that $D_r(z) \subseteq \Omega$.

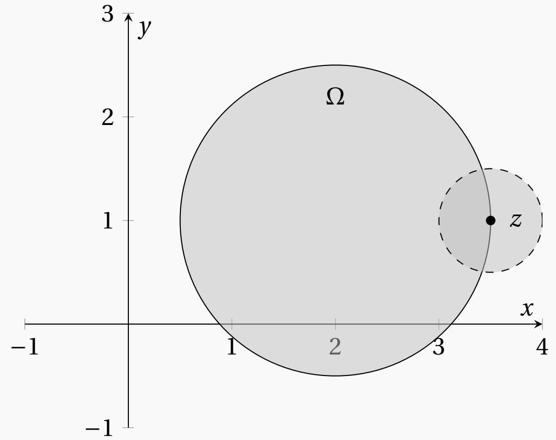
Example 2.7

Visually, we have



We can see that we can find a disk $D_r(z)$ completely contained within Ω.

A point that is not an interior point can be found with a closed disk:



A point z on the boundary will always define a disk $D_r(z)$ that contains points outside of Ω, for any r we choose.

Definition 2.8: Open set

A set $\Omega \subseteq \mathbb{C}$ is called an *open set* if every point in Ω is an interior point.

Example 2.9

From the previous example, we can see that an open disk $D_r(z)$ is an open set, whereas the closed disk $\overline{D_r(z)}$ is not an open set.

The proof is omitted, but can be shown through the triangle inequality.

For more notation, we define $\Omega^\circ \subseteq \Omega$ as the set of all interior points in Ω. As such, Ω is an open set if and only if $\Omega^\circ = \Omega$.

Definition 2.10: Closed set

A set $\Omega \subseteq \mathbb{C}$ is a *closed set* if $\Omega^c = \{z \in \mathbb{C} \mid z \notin \Omega\}$ is an open set.

Theorem 2.11: Closed Disks are Closed Sets

Any closed disk is a closed set.

Proof. Consider the closed disk $\overline{D_R(z_0)}$ (abbreviated as $\overline{D}$ here for convenience). To show that $\overline{D}$ is a closed set, we need to show that for any $z \notin \overline{D}$, there exists some $r > 0$ such that $D_r(z) \subseteq \overline{D}^c$.

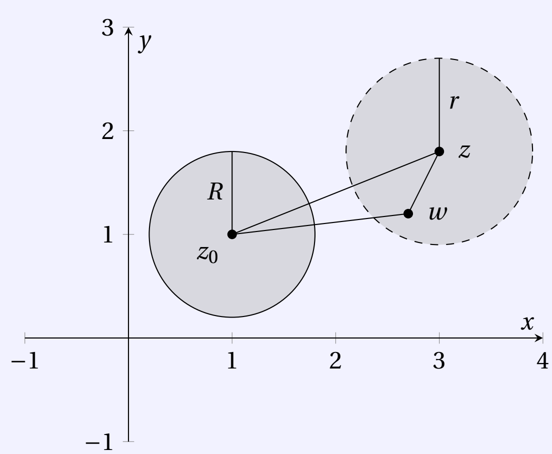
Firstly, we claim that $|z_0 - z| > R$, since $z \notin \overline{D}$. As such, let us take $0 < r < |z_0 - z| - R$.

Our claim is that $D_r(z) \subseteq \overline{D}^c$. Suppose we pick any point $w \in D_r(z)$; it is sufficient to check $|w - z_0| > R$, since this is the definition of $\overline{D}^c$.

This follows from the triangle inequality:

$$\begin{aligned}
|w - z_0| &= |(w - z) + (z - z_0)| \\
&\geq |z - z_0| - |w - z| \\
&> |z - z_0| - r && (|w - z| < r) \\
&> |z - z_0| - (|z - z_0| - R) && (r < |z - z_0| - R) \\
&= R
\end{aligned}$$

This means that $w \in \overline{D}^c$, or equivalently $w \notin \overline{D}$



□

2.4 Limit points

Definition 2.12: Limit point

For a set $\Omega \subseteq \mathbb{C}$, a point $w \in \mathbb{C}$ is a *limit point* of Ω if there is some $\{z_n\}$ in Ω with every $z_n \neq w$ such that $z_n \to w$ as $n \to \infty$.

Note that we require $z_n \neq w$ for all n, because the trivial case of $z_n = w$ for all N makes every point $w \in \Omega$ a limit point.

Example 2.13

If we take w, we can consider disks of decreasing radii $D_{r_n}(w)$; taking the sequence of points $z_n \in D_{r_n}(w)$, we can see that $\{z_n\} \to w$. It turns out that any point in an open disk is a limit point of the open disk.

Another example is with a closed disk; a boundary point will have a sequence $\{z_n\}$ along the radius that converges to w. The same can be said with any interior point. As such, any point in a closed disk is a limit point of the closed disk, and in fact any limit point will be contained in the closed disk.

One other example is with a boundary point w of an open disk. Notice that $w \notin \Omega$, but will still have a sequence $\{z_n\}$ in Ω that converges to w. As such, not every limit point of an open disk will be contained in the open disk.

Notationally, we introduce $\overline{\Omega}$ as the *closure* of Ω, defined by $\overline{\Omega} = \Omega \cup \{\text{limit points of } \Omega\}$. Sometimes the set of all limit points of Ω is denoted as Ω'.

Theorem 2.14: Closed Sets and Limit Points

$\Omega \subseteq \mathbb{C}$ is closed if and only if Ω contains all of its limit points. Equivalently, Ω is closed if and only if $\Omega = \overline{\Omega}$.

As a consequence, $\overline{D_r(z_0)}$ is the closure of the open disk $D_r(z_0)$.

Definition 2.15: Boundary points

The boundary points of a set Ω is $\partial\Omega = \overline{\Omega} \setminus \Omega^\circ$.

Example 2.16

For a disk,

$$\partial D_r(z_0) = \overline{D_r(z_0)} \setminus D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\},$$

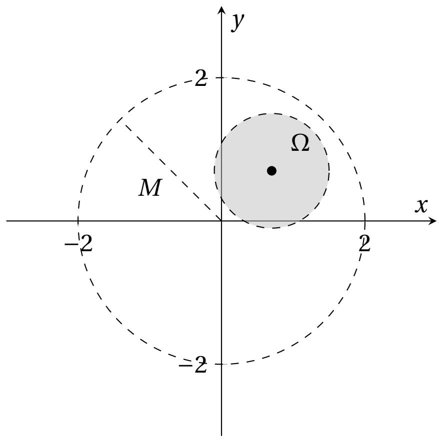
or the circle of radius r centered at z_0.

2.5 Boundedness

Definition 2.17: Bounded

A set $\Omega \subseteq \mathbb{C}$ is *bounded* if there is some $M > 0$ such that every $z \in \Omega$ satisfies $|z| < M$.

Visually, we're taking a disk centered at the origin that contains Ω:



Example 2.18

$\overline{D_r(z_0)}$ and $D_r(z)$ are both bounded, but $\mathbb{C}$ itself is not bounded.

Similarly, $\mathbb{R} \subseteq \mathbb{C}$ and $i\mathbb{R} \subseteq \mathbb{C}$ (the real and imaginary axes of $\mathbb{C}$ respectively) are not bounded.

Definition 2.19: Diameter

For a bounded set $\Omega \subset \mathbb{C}$, we define the diameter of Ω as

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z-w|.$$

Note that $\operatorname{diam}(\Omega) < \infty$, since Ω is bounded.

Definition 2.20: Compact

A set $\Omega \subseteq \mathbb{C}$ is a *compact* set if any open cover of Ω has a finite subcover.

Theorem 2.21: Heine–Borel Theorem

A set is compact if and only if it is closed and bounded.

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Lecture 3

Sequentially Compact Sets, Continuity

Definition 3.1: Sequentially Compact

A set $\Omega \subseteq \mathbb{C}$ is sequentially compact if any sequence $\{z_n\}$ in Ω has a convergent subsequence in Ω (i.e. its limit is in Ω).

Theorem 3.2

In any metric space, a set Ω is compact if and only if it is sequentially compact.

As such, we only need to show a set Ω is closed and bounded in order for us to conclude that it is compact and sequentially compact—closed and bounded sets have the same properties.

3.1 Sequences of Subsets

Theorem 3.3: Sequences of subsets

Suppose we have a sequence of subsets $\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \cdots$, where each Ω_i is a nonempty compact subset of $\mathbb{C}$. Further, suppose $\lim_{n\to\infty} \operatorname{diam}(\Omega_n) = 0$.

Then, the intersection of this sequence of compact sets is not empty but contains a unique point.

Proof. Suppose we take a point $z_n \in \Omega_n$ for each $n = 1, 2, \ldots$; this creates a sequence $\{z_n\}$ in $\mathbb{C}$.

Since $\lim_{n\to\infty} \operatorname{diam}(\Omega_n) = 0$, then $|z_m - z_n| \to 0$, since this distance is bounded by the diameter of $\Omega_{\max(m,n)}$, and this diameter converges to zero. Formally, for any ε we can always find an N such that $m, n > N \implies \operatorname{diam}(\Omega_{\max(m,n)}) < \varepsilon$, making $|z_m - z_n| < \varepsilon$ as well. As such, $\{z_n\}$ is a Cauchy sequence.

This means that $\{z_n\}$ is a convergent sequence (since $\mathbb{C}$ is complete), and $z_n \to z_0$ as $n \to \infty$.

We know that each Ω_k is a compact set, and thus also sequentially compact. This means that $\{z_n\}$ is a convergent sequence in Ω_k and its limit is also contained in Ω_k. This limiting point $z_0 \in \Omega_k$ for all k, and thus $z_0 \in \bigcap_{n=1}^{\infty} \Omega_n$.

We've just proven that the intersection is nonempty; we now need to show that it contains only one unique point.

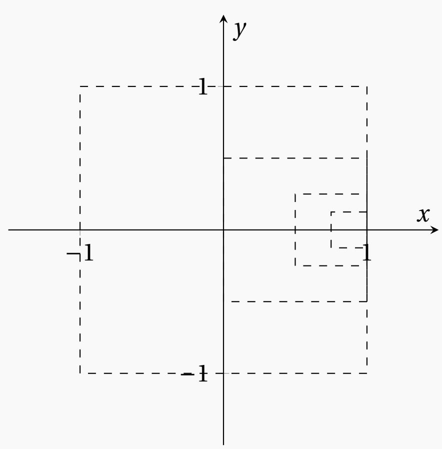
Let us take two points $z_0, z_0' \in \bigcap_{n=1}^{\infty} \Omega_n$; we'll show that it must be the case that $z_0 = z_0'$.

Let us consider any Ω_n; we know that $z_0, z_0' \in \Omega_n$. By definition, we must also have $\left|z_0 - z_0'\right| \le \operatorname{diam}(\Omega_n)$. Taking $n \to \infty$, we have $\operatorname{diam}(\Omega_n) \to 0$, and thus $\left|z_0 - z_0'\right| = 0$. Since $\mathbb{C}$ is a metric space, this means that $z_0 = z_0'$ (two points with zero distance must be equal). □

Example 3.4

As an example, suppose we have $\{\Omega_n \coloneqq \overline{D_{\frac{1}{n}}(0)}\}$. Here, we have $\bigcap_{n=1}^{\infty} \Omega_n = 0$. This is because any point $z \neq 0$ will always be excluded by some disk $D_{\frac{1}{n}}(0)$.

However, if we consider the sequence of open boxes:



Any point we choose will be excluded by some smaller box, and since we take open sets, the boundaries are also excluded. This means that the intersection of all the open boxes is empty.

3.2 Connectedness

Definition 3.5: Connected

An open subset $\Omega \subseteq \mathbb{C}$ is *connected* if there is no way to write Ω as a disjoint union of two nonempty open sets.

(This can be generalized to any subset of $\mathbb{C}$.)

Example 3.6

Let us take an open disk Ω. Is it possible for $\Omega = U \cup V$ such that $U \cup V = \varnothing$ and $U, V \neq \varnothing$ and U, V are open?

Intuitively, if we try to just cut the disk into two parts by a line, we'd find that the points along the boundary are not included in either U or V. This is because open sets cannot contain boundary points (otherwise they would not be open), and thus the union is not Ω.

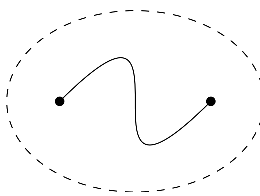
It turns out that there is no way for us to split Ω like this—the open disk is connected.

Definition 3.7: Path-connected

An open subset $\Omega \subseteq \mathbb{C}$ is *path-connected* if any two points $z_0, z_1 \in \Omega$ can be joined by a continuous path in Ω.

(This can be generalized to any subset of $\mathbb{C}$.)

What does it mean for a continuous path in Ω to join two points?



More rigorously, we can find a continuous map $s : [0,1] \to \Omega$ such that $s(0) = z_0$ and $s(1) = z_1$. Alternatively, we can think of the domain $[0,1]$ as time; we start at z_0 and continuously move in Ω and end at z_1.

Intuitively, the open disk is path-connected, and two disjoint open disks are not path-connected (the two disjoint disks are not connected either).

For all open sets in $\mathbb{C}$, path connectedness and connectedness are synonymous; any open set that is connected is also path-connected, and vice versa. This is not true in general, but true for open sets in $\mathbb{C}$.

Definition 3.8: Region

An open connected (equiv. path-connected) subset of $\mathbb{C}$ is called a *region*.

3.3 Functions on $\mathbb{C}$

For a subset $\Omega \subseteq \mathbb{C}$, we will consider complex-valued functions defined on Ω. That is, we will look at functions $f : \Omega \to \mathbb{C}$.

Any point $z \in \Omega \subseteq \mathbb{C}$ in the domain of f can be written as $z = x + iy$, which is equivalent to a point $(x, y) \in \mathbb{R}^2$. The same equivalence can be said for any point in the codomain of f.

This means that $f(z) = \text{Re}(f(z)) + i\,\text{Im}(f(z))$; letting $u(z) = \text{Re}(f(z))$ and $v(z) = \text{Im}(f(z))$, we now have the pair of real-valued functions $u, v : \Omega \to \mathbb{R}$ (equivalently the domain can be seen as equivalent to $\mathbb{R}^2$, so we have $(u, v) : \mathbb{R}^2 \to \mathbb{R}^2$). As such, any complex-valued function can be equivalently considered as a pair of real-valued functions.

3.3.1 Continuity

Definition 3.9: Continuous

A function $f : \Omega \to \mathbb{C}$ is continuous at $z_0 \in \Omega$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \implies \left|f(z) - f(z_0)\right| < \varepsilon.$$

Further, a function $f : \Omega \to \mathbb{C}$ is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Notationally, if f is continuous on Ω, we notate as $f \in C^0(\Omega)$.

More generally, from 104, continuity of a map from any metric space X to any metric space Y can be characterized similarly; we just use the distance function d_X on the LHS and use d_Y on the RHS.

Theorem 3.10

If f is continuous at z_0, then for any sequence $\{z_n\}$, we have $z_n \to z_0 \implies f(z_n) \to f(z_0)$.

We can equivalently characterize continuity in terms of the pair of real-valued parts of the complex numbers; if f is continuous at $z_0 = x_0 + iy_0$, then $u = \text{Re}(f)$ and $v = \text{Im}(f)$ are continuous at (x_0, y_0).

If we consider functions $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, continuity can be characterized similarly, since we have individual metrics on $\mathbb{C}$, and as such we have a metric on $\mathbb{C}$. We can show that addition, multiplication, division (excluding division by zero) are all continuous functions from $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

As such, taking two functions $f, g : \Omega \to \mathbb{C}$, we can look at $(f + g)(z) = f(z) + g(z)$. If f, g are both continuous, then $f + g$ is also continuous.

We can think of this as a composition of two mappings:

$$\Omega \xrightarrow{(f,g)} \mathbb{C} \times \mathbb{C} \xrightarrow{+} \mathbb{C}.$$

As such, $f + g$ is the composition of two continuous functions. We can then work out that the composition of two continuous functions is also continuous, and the continuity of addition follows (and subtraction, multiplication, division as well, as they're all characterized in a similar way through composition).

Theorem 3.11

Let $\Omega \subseteq \mathbb{C}$ be a compact subset, with $f \in C^0(\Omega)$. Further, let $u = \text{Re}(f)$ and $v = \text{Im}(f)$.

Then, u, v will always achieve its maximum and minimum on Ω.

In particular, since norms are continuous, and $|u|$ and $|v|$ are both continuous, then $\left|f\right|$ is also continuous on Ω and $\left|f\right|$ always achieves its maximum and minimum on Ω.

3.4 Holomorphic Functions

Everything covered prior from now can be equivalently stated for $\mathbb{R}^2$, but now we'll talk about something unique to $\mathbb{C}$.

Definition 3.12: Holomorphic Function

Suppose we take an open set $\Omega \subset \mathbb{C}$ and a function $f : \Omega \to \mathbb{C}$.

f is *holomorphic* (also called *complex differentiable*) at $z_0 \in \Omega$ if

$$f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists for $h \in \mathbb{C}$ such that $z_0 + h \in \Omega$.

We define this limit to be the *derivative* of f at z_0, denoted by $f'(z_0) \in \mathbb{C}$.

Division on $\mathbb{C}$ is very different from division on $\mathbb{R}$, so holomorphic functions should naturally be quite different from differentiable functions on $\mathbb{R}$.

Example 3.13

As an example, let us take $f : \mathbb{C} \to \mathbb{C}$, such that $f : z \mapsto z$.

We can see that f is holomorphic at every $z_0 \in \mathbb{C}$, and in fact $f'(z_0) = 1$.

If we take any $z_0 \in \mathbb{C}$, we have

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{(z_0+h) - (z_0)}{h} = \frac{h}{h} = 1,$$

for any $h \in \mathbb{C}$. As such, the limit is 1, and f is holomorphic at all points, with a derivative $f'(z) = 1$.

1/27/2022

Lecture 4

Holomorphic Functions

Example 4.1

Let us take $f(z) = z^2$; what is $f'(z_0)$?

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} &= \frac{(z_0+h)^2 - z_0^2}{h} \\ &= \frac{(z_0+h+z_0)(z_0+h-z_0)}{h} \\ &= 2z_0 + h\end{aligned}$$

As such,

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h\to 0}(2z_0 + h) = 2z_0.$$

Example 4.2

We can similarly check that if $f(z) = z^3$, then $f'(z) = 3z^2$; the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ may help.

In general, suppose we have a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

with $a_0, \dots, a_n \in \mathbb{C}$ and $a_n \neq 0$, $n \geq 0$.

We then have

$$p'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1}.$$

As such, $p(z)$ is holomorphic on $\mathbb{C}$.

Definition 4.3: Entire function

An *entire function* is a function that is holomorphic on $\mathbb{C}$.

From before, we can see that any polynomial is an entire function.

What if we have negative integer exponents?

Example 4.4

Suppose we have $f(z) = z^{-1} = \frac{1}{z}$. f doesn't make sense when $z = 0$, so the domain of f is $\mathbb{C} \subseteq \{0\}$, which we will define as $\mathbb{C}^*$. (Note that $\{0\}$ is closed in $\mathbb{C}$, so $\mathbb{C}^*$ is open in $\mathbb{C}$.)

We have for $z_0 \in \mathbb{C}^*$

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} &= \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} \\ &= \frac{\frac{z_0(z_0+h)}{z_0-(z_0+h)}}{h} \\ &= -\frac{1}{z_0(z_0+h)}\end{aligned}$$

As such,

$$\lim_{h\to 0} -\frac{1}{z_0(z_0+h)} = -\frac{1}{z_0^2}.$$

This means that $f'(z) = -\frac{1}{z^2}$.

We can see that the power rule applies to complex functions as well; $f(z) = z^n$ means $f'(z) = nz^{n-1}$.

Example 4.5

We already know that $f(z) = z$ is holomorphic; what about $f(z) = \overline{z}$? Geometrically, this is a reflection across the real axis. We claim that this function is *not* holomorphic.

We have

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} &= \frac{\overline{z_0+h} - \overline{z_0}}{h} \\ &= \frac{\overline{z_0} + \overline{h} - \overline{z_0}}{h} && \text{(conjugate is linear)} \\ &= \frac{\overline{h}}{h}\end{aligned}$$

Now the question is: does $\lim_{h\to 0} \frac{\overline{h}}{h}$ exist?

We can write $h = h_x + i h_y$, with $h_x, h_y \in \mathbb{R}$. Saying that $h \to 0$ is equivalent to saying $(h_x, h_y) \to (0,0) \in \mathbb{R}^2$.

This is a limit we've considered before in multivariable calculus. There are several directions in which h can go to zero—from any line, from an axis, or even spiraling.

In order for the limit to exist, the limit must be defined no matter which direction we go in.

Suppose we take $h \to 0$ along the real axis. In this situation, we have $\overline{h} = h$, meaning $\frac{\overline{h}}{h} = 1$. This makes $\lim_{h\to 0} \frac{\overline{h}}{h} = 1$.

Suppose we take $h \to 0$ along the imaginary axis. In this situation, we have $\overline{h} = -h$, meaning $\frac{\overline{h}}{h} = -1$. This makes $\lim_{h\to 0} \frac{\overline{h}}{h} = -1$.

Since these two limits have different values, we can conclude that the limit doesn't exist—f is not holomorphic at *any* $z_0 \in \mathbb{C}$.

Example 4.6

Let us consider $f(z) = |z|^2$. Since this function is equivalent to $f(z) = z\overline{z}$, the appearance of $\overline{z}$ makes the function not holomorphic.

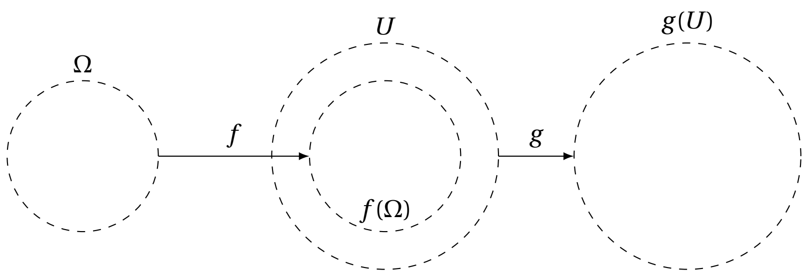
4.1 Properties of Holomorphic Functions

Suppose f, g are two holomorphic functions on some open set $\Omega \subseteq \mathbb{C}$. (Equivalently we can consider this pointwise for some $z_0 \in \Omega$)

- $f \pm g$ is also holomorphic, with $(f \pm g)' = f' \pm g'$.
- $f \cdot g$ is also holomorphic, with $(f \cdot g)' = f' \cdot g + f \cdot g'$.
- If $g \neq 0$ (or $g(z_0) \neq 0$), then $\frac{f}{g}$ is also holomorphic, with $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$.

So far, these have been identical to real functions.

Suppose f is holomorphic on $\Omega \subseteq \mathbb{C}$ and g is holomorphic on $U \subseteq \mathbb{C}$, where both Ω and U are open in $\mathbb{C}$. Suppose we also have $U \subseteq f(\Omega)$.



We can conclude that $g \circ f = g(f(z))$ is also holomorphic on Ω, with $(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$ for $z \in \Omega$. This is called the *chain rule*.

Theorem 4.7: Holomorphic functions are continuous

If $f'(z_0)$ exists, then f is continuous at z_0.

Proof. If $f'(z_0)$ exists, then
$$\frac{f(z_0 + h) - f(z_0)}{h} \to f'(z_0) \text{ as } h \to 0.$$
We can write this limit as an equality with a perturbation:
$$\frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) + \psi(z_0, h).$$

Here, we have $\psi(z_0,h)\to 0$ as $h\to 0$; multiplying by h and expanding, we end up with

$$f(z_0+h)-f(z_0)=hf'(z_0)+h\psi(z_0,h).$$

Since $f'(z_0)$ is constant, and $\psi(z_0,h)$ also goes to zero as $h\to 0$, then we end up with the fact that the entire RHS goes to zero as $h\to 0$.

As such, $\lim_{h\to 0} f(z_0+h)=f(z_0)$, which is the definition of continuity at z_0. □

4.1.1 Holomorphic Functions as a Map

We've considered before $f:\ \Omega\to\mathbb{C}$ with $\Omega\subseteq\mathbb{C}$; we can see that both the domain and codomain can be equivalently represented as $U\subseteq\mathbb{R}^2$ and $\mathbb{R}^2$ respectively, through the imaginary and real parts of f, represented by $u(z)=\operatorname{Re}(f(z))$ and $v(z)=\operatorname{Im}(f(z))$.

The congruent mapping $F:\ U\to\mathbb{R}^2$ is $F:\ (x,y)\mapsto(u(x,y),v(x,y))$.

If we consider f as a holomorphic function, what properties does F have?

Suppose we take $z_0\in\Omega$, where $z_0=x_0+iy_0$, where f is holomorphic at z_0. We want to determine properties of $u(x,y)$ and $v(x,y)$ at (x_0,y_0).

Since f is holomorphic at z_0, we have (with $h=h_1+ih_2$)

$$\frac{f(z_0+h)-f(z_0)}{h}=\frac{u(x_0+h_1,y_0+h_2)-u(x_0,y_0)+i\big(v(x_0+h_1,y_0+h_2)-v(x_0,y_0)\big)}{h_1+ih_2}$$

Multiplying by the conjugate of the denominator, we have

$$\begin{aligned}
&=\frac{u(x_0+h_1,y_0+h_2)-u(x_0,y_0)+i\big(v(x_0+h_1,y_0+h_2)-v(x_0,y_0)\big)(h_1-ih_2)}{h_1^2+h_2^2}\\
&=\frac{(u(x_0+h_1,y_0+h_2)-u(x_0,y_0))\cdot h_1+(v(x_0+h_1,y_0+h_2)-v(x_0,y_0))\cdot h_2}{h_1^2+h_2^2}\\
&\quad+i\frac{(-u(x_0+h_1,y_0+h_2)-u(x_0,y_0))\cdot h_2+(v(x_0+h_1,y_0+h_2)-v(x_0,y_0))\cdot h_1}{h_1^2+h_2^2}
\end{aligned}$$

If we were to look at the limit, the limit should always exist no matter which direction we take, and all limits should agree (since f is holomorphic).

As such, suppose we take the limit along the real axis; that is, $h_2=0$. We then have

$$\lim_{\substack{h_1\to 0\\ h_2=0}}\frac{f(z_0+h)-f(z_0)}{h}=\frac{u(x_0+h_1,y_0)-u(x_0,y_0)}{h_1}+i\frac{v(x_0+h_1,y_0)-v(x_0,y_0)}{h_1}=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0).$$

As such, $f'(z_0)$ exists means that $\frac{\partial u}{\partial x}(x_0,y_0)$ and $\frac{\partial v}{\partial x}(x_0,y_0)$ both exist, with

$$f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0).$$

Equivalently, we can take the limit along the imaginary axis; that is, $h_1=0$. The same calculation will give us

$$f'(z_0)=\frac{\partial v}{\partial y}(x_0,y_0)-i\frac{\partial u}{\partial y}(x_0,y_0).$$

As such, we have $\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)-i\frac{\partial u}{\partial y}(x_0,y_0)$. The real and imaginary parts must match up, so the existence of f' implies the following system of partial differential equations:

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\[2mm] \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases}$$

The conclusion here is that if f is holomorphic at z_0, then the above system holds at (x_0, y_0). Similarly, if f is holomorphic on Ω, then the above system holds on $U \subseteq \mathbb{R}^2$.

The system of partial differential equations is also called the Cauchy-Riemann equations.

Example 4.8

Let us take $f(z) = \overline{z}$. Splitting this into its real and imaginary parts, we have $u(x, y) = x$ and $v(x, y) = -y$.

The partial derivatives are $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, and $\frac{\partial v}{\partial y} = -1$.

The Cauchy-Riemann equations requires $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, but $\frac{\partial u}{\partial x} = 1$ where $\frac{\partial v}{\partial y} = -1$, so the first equation does not hold. It also requires $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and since $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$, the second equation does hold.

However, since the first equation does not hold, this pair of functions u, v does not satisfy the Cauchy-Riemann equations.

4.2 Complex Partial Derivatives

Suppose we have $f : \Omega \to \mathbb{C}$. We can consider f as a function of two inputs: $f(z, \overline{z})$.

We define

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

The Cauchy-Riemann equations are then equivalent to saying that $\frac{\partial f}{\partial \overline{z}} = 0$. As such, $f'(z) = \frac{\partial f}{\partial z}$.

We can check this conclusion;

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} = 0 &\iff \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = 0 \\
&\iff \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \\
&\iff \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y} \end{cases}
\end{aligned}$$

In the last step, we equated both the real and imaginary parts.

2/1/2022

Lecture 5

More Holomorphic Functions

We talked about the Cauchy-Riemann equations last time, and implications if $f : \Omega \to \mathbb{C}$ is holomorphic. It turns out the reverse direction is not necessarily true.

Example 5.1

Suppose we have $f : \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = \sqrt{|\mathrm{Re}(z)| \cdot |\mathrm{Im}(z)|}.$$

If we write $f(z) = u(z) + i\,v(z)$, we can show that all partial derivatives of u, v with respect to x, y are all 0 at $(0, 0)$.

As such, this satisfies the Cauchy-Riemann conditions trivially—but f is *not* holomorphic at $(0,0)$. (The limits along $x = y$ and $x = -y$ are different.)

As such, we need to modify our conditions in order to prove the reverse direction.

Definition 5.2: Multivariate Differentiability

A function $F = (u, v) : U \to \mathbb{R}^2$ is *differentiable* at (x_0, y_0) if there is a 2×2 matrix $\mathbf{J}$ such that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\left| F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) - \mathbf{J}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right|}{\left| \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right|} = 0,$$

where $|\cdot|$ of a vector denotes its Euclidean norm $\left| \begin{bmatrix} x \\ y \end{bmatrix} \right| = \sqrt{x^2 + y^2}$.

Firstly, differentiability only requires the existence of $\mathbf{J}$; but what is the value of $\mathbf{J}$?

It turns out that if we take $h_1 = 0$ or $h_2 = 0$, the limit just turns into the definition of partial derivatives of each component of F.

As such, if F is differentiable at (x_0, y_0), then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at (x_0, y_0).

Further, $\mathbf{J}$ is exactly the Jacobian matrix at (x_0, y_0):

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\Bigg|_{(x_0,y_0)}.$$

With our earlier example, we can see that $f(z) = \sqrt{|\text{Re}(z)| \cdot |\text{Im}(z)|}$ is not differentiable at $(0,0)$, and as such this stronger condition excludes this function.

We will now state the stronger theorem:

Theorem 5.3: Cauchy-Riemann equations and holomorphicity

Suppose we have $f : \Omega \to \mathbb{C}$, with the equivalent representation $F : U \to \mathbb{R}^2$, as defined earlier.

f is holomorphic at z_0 if and only if F is differentiable at (x_0, y_0) and the following system holds (Cauchy-Riemann Equations):

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Proof. ($\Longrightarrow$) We've already proven a weaker conclusion before, showing that if f is holomorphic, the partial derivatives of u, v exist, and the Cauchy-Riemann equations hold. As such, all that is left is to show that F is differentiable at (x_0, y_0).

Since f is holomorphic at z_0, we have

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=f'(z_0).$$

Equivalently, we can say that $\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)$ goes to zero as $z\to z_0$. Simplifying, we have

$$\frac{f(z)-f(z_0)-f'(z_0)(z-z_0)}{z-z_0}\to 0$$
$$\left|\frac{f(z)-f(z_0)-f'(z_0)(z-z_0)}{z-z_0}\right|\to 0$$
$$\frac{|f(z)-f(z_0)-f'(z_0)(z-z_0)|}{|z-z_0|}\to 0$$

Here, we can see that $\left\|\begin{bmatrix}h_1\\h_2\end{bmatrix}\right\|\equiv|z-z_0|$, since h_1 and h_2 correspond to the real and imaginary parts. Further, something similar happens in the numerator with $F(x_0+h,y_0+h_2)-F(x_0,y_0)$ corresponding to $f(z)-f(z_0)$. As such, we just need to calculate $\operatorname{Re}(f'(z_0)(z-z_0))$ and $\operatorname{Im}(f'(z_0)(z-z_0))$.

Splitting into the real and imaginary parts, we have

$$\begin{aligned}\operatorname{Re}(f'(z_0)(z-z_0))&=\operatorname{Re}\left(\left(\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}\right)(h_1+ih_2)\right)\\&=\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}h_1-\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}h_2\end{aligned}$$

Notice that if we multiply out $\mathbf{J}\begin{bmatrix}h_1\\h_2\end{bmatrix}$, we have

$$\mathbf{J}\begin{bmatrix}h_1\\h_2\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x}h_1+\frac{\partial u}{\partial y}h_2\\\frac{\partial v}{\partial x}h_1+\frac{\partial v}{\partial y}h_2\end{bmatrix}.$$

The first element of the resulting vector is equal to what we just got, utilizing the Cauchy-Riemann equations to say $-\left.\frac{\partial v}{\partial x}\right|_{(x0,y_0)}=\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}$.

The imaginary part proceeds similarly; we have

$$\operatorname{Im}(f'(z_0)(z-z_0))=\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}h1+\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}h_2.$$

As such, we've just shown equivalence to all the parts of the definition of differentiability of F at (x_0,y_0), and thus proved the forward direction.

($\Longleftarrow$) We can reuse some of these calculations. If F is differentiable at (x_0,y_0), then $\mathbf{J}$ exists consisting of the partial derivatives of F. Further, we know that the Cauchy-Riemann equations hold.

Doing the steps earlier in reverse, we'd end up with the fact that

$$\left|\frac{f(z)-f(z_0)-f'(z_0)(z-z_0)}{z-z_0}\right|\to 0 \text{ as } z\to z_0.$$

This is again the definition of a holomorphic function, and as such f is holomorphic at z_0. □

As a remark, we've just shown that if f is holomorphic, then the partial derivatives of $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ exist. However, later we'll prove that any order of partial derivatives also exist—that is, we'll show later the smoothness of $\text{Re}(f)$ and $\text{Im}(f)$.

Definition 5.4: Harmonic function

A function $u(x, y)$ is a *harmonic function* if the Laplacian is zero:

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Corollary 5.5

It turns out that if $f = u + iv$ is holomorphic, then u, v are *harmonic* functions.

Proof. Looking at $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

In this case, we have $\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$ since v is regular (which we haven't proved yet), and as such

$$\Delta u = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0,$$

as desired. □

We've talked about the Jacobian matrix of $f = u + iv$:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

If we look at the determinant, we have

$$\det \mathbf{J} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

If f is holomorphic, then we can use the Cauchy-Riemann equations to simplify

$$\begin{aligned}\det(\mathbf{J}) &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \geq 0\end{aligned}$$

As such, we have another (necessary but not sufficient) condition for whether a function f is holomorphic; if $\det(\mathbf{J}_f) < 0$, then f is not holomorphic.

Example 5.6

If we have $f(z) = \bar{z}$, we have $u(x,y) = x$ and $v(x,y) = -y$. The Jacobian is then

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The determinant is thus $\det(\mathbf{J}) = -1 < 0$, and as such f is not holomorphic.

5.1 Power Series

Recall that the simplest holomorphic functions are polynomials (of degree n)

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where $a_i \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$, and $a_n \neq 0$.

A power series is a generalization of polynomial functions—we don't stop at a finite degree; we have an infinite sum

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k \in \mathbb{C}$.

If this series is convergent, then it can be considered as a function in z. If the series is divergent, we call the series "formal power series", but we don't study divergent series too much in this course—we mainly focus on conditions that make power series converge.

Example 5.7

Suppose we have

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots.$$

Note that if $z \in \mathbb{R}$, then this series is convergent for any $z \in \mathbb{R}$. This result can be generalized to the complex numbers; the series is convergent for any $z \in \mathbb{C}$ as well.

Further, this is the exponential function of z, i.e.

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Example 5.8

Suppose we have

$$\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots.$$

Note that the ratio between successive terms is fixed at z; this is a geometric series.

As such, one question to ask is: for what z is this series convergent?

Looking at the partial sums $S_N = \sum_{k=0}^{N} z^k = 1 + z + \cdots + z^N$, we have

$$\begin{aligned} S_N &= 1 + z + \cdots + z^{N-1} + z^N \\ z \cdot S_N &= z + z^2 + \cdots + z^N + z^{N+1} \end{aligned}$$

$$S_N - z \cdot S_N = 1 - z^{N+1}$$
$$(1-z)S_N = 1 - z^{N+1}$$
$$S_N = \frac{1-z^{N+1}}{1-z}$$

In the last equality, we assume $z \neq 1$. We now ask whether $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z}$ exists (for $z \neq 1$).

Everything in the fraction is constant except for z^{N+1}; as such, $z^{N+1} \to 0$ as $N \to \infty$ if $|z| < 1$, and we can show that z^{N+1} is divergent if $|z| \geq 1$.

In particular, if $z = 1$, the formula doesn't make sense, but the partial sum is easy to compute—every term is 1, so the partial sum is $S_N = N + 1$, which is divergent as $N \to \infty$.

Thus, the geometric series is convergent if and only if $|z| < 1$.

Further, we can compute what the sum is—since z^{N+1} goes to zero when the series is convergent, the limit turns out to be

$$\sum_{k=0}^{\infty} z^k = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z}.$$

Let us consider a power series $\sum_{n=0}^{\infty} a_n z^n$.

We have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \in \overline{\mathbb{R}},$$

where $\overline{\mathbb{R}}$ is the extended real numbers (including ∞ and $-\infty$).

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Lecture 6

Power Series

Definition 6.1: Convergence Radius

We define the *convergence radius* of a power series as

$$R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Further, when the denominator is zero, we define $R = +\infty$, and when the denominator is $+\infty$, then we define $R = 0$.

Example 6.2

Last time, we looked at

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

It turns out that this series converges at all complex numbers; we can look at the convergence radius:

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}}} = \frac{1}{\limsup \frac{1}{\sqrt[n]{n!}}}.$$

As such, we just need to figure out $\limsup_{n\to\infty} \sqrt[n]{n!}$. Intuitively, just trying out values can tell us that the limit is $+\infty$.

Formally, we have

$$\begin{aligned} n! &= n(n-1)(n-2)(n-3)\cdots(3)(2)(1) \\ &\geq n(n-1)(n-2)(n-3)\cdots(\frac{n}{2}) \\ &\geq \frac{n}{2}\cdot\frac{n}{2}\cdot\frac{n}{2}\cdots\frac{n}{2} \\ &= \left(\frac{n}{2}\right)^{\frac{n}{2}} \end{aligned}$$

As such,

$$\sqrt[n]{n!} \geq \sqrt{\left(\frac{n}{2}\right)^{\frac{n}{2}}} = \sqrt{\frac{n}{2}} \to \infty.$$

Going back to the original convergence radius, we have

$$R = \frac{1}{\limsup_{n\to\infty} \frac{1}{\sqrt[n]{n!}}} \to \frac{1}{\frac{1}{\infty}} = +\infty.$$

As such, the series converges on the disk centered at 0 with radius $+\infty$. This disk contains the entire complex plane (i.e. the disk is $\mathbb{C}$), so this series converges for all complex numbers.

Example 6.3

Last time, we also looked at the geometric series

$$\sum_{n=0}^{\infty} z^n.$$

We calculated the partial sums of the series, and determined that if $|z| < 1$, then the series is convergent, and when $|z| > 1$, then it is divergent.

Because of this, we should get a convergence radius of $R = 1$. Doing the calculations, we have

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{1}} = 1.$$

Let's now give a statement for the general case; but before then, we need to establish a notion of absolute convergence.

Definition 6.4: Absolute convergence

A series $\sum_{n=0}^{\infty} a_n z^n$ *converges absolutely* if $\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n||z|^n$ converges.

Lemma 6.5

If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely, then it converges.

Proof. Recall that we say that a series is convergent if the series of partial sums converges.

As such, we define $S_N = \sum_{n=0}^{N} a_n z^n$, which forms a sequence $\{S_N\}_{N=0}^{\infty}$, which we want to show is convergent.

We further define $\tilde{S}_N = \sum_{n=0}^{N} |a_n z^n|$, which we know converges—that is, for any $\varepsilon > 0$ there exists an $M > 0$ such that $N > M$ and $p \geq 0$ implies $\left|\tilde{S}_{N+p} - \tilde{S}_n\right| < \varepsilon$.

We have

$$\begin{aligned}
\left|S_{N+p} - S_N\right| &= \left|\sum_{n=0}^{N+p} a_n z^n - \sum_{n=0}^{N} a_n z^n\right| \\
&= \left|\sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{N+p} a_n z^n - \sum_{n=0}^{N} a_n z^n\right| \\
&= \left|\sum_{n=N+1}^{N+p} a_n z^n\right| \\
&\le \sum_{n=N+1}^{N+p} \left|a_n z^n\right| && \text{(triangle inequality)} \\
&= \left|\sum_{n=0}^{N+p} \left|a_n z^n\right| - \sum_{n=0}^{N} \left|a_n z^n\right|\right| \\
&\le \varepsilon
\end{aligned}$$

The last inequality is because the expression is exactly $\left|\tilde{S}_{N+p} - \tilde{S}_N\right|$.

As such, absolute convergence implies convergence. □

Theorem 6.6: Root test

Suppose we consider the power series $\sum_{n=0}^{\infty} a_n z^n$. We define $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$.

We claim that

- For any z with $|z| < R$, the series converges absolutely.
- For any z with $|z| > R$, the series diverges.

Proof. First, suppose we have $|z| < R$. That is,

$$|z| < R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

There are three cases of R that we'd need to consider separately: $R = 0$, $R = +\infty$, and $R \in (0, +\infty)$; we will only consider the last case here.

Multiplying by the limsup, we have

$$|z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1.$$

Suppose for now that we don't have the limsup. Since the reals are dense, we can always find an $r \in (0,1)$ such that

$$|z| \sqrt[n]{|a_n|} < r < 1 \implies |z|^n |a_n| < r^n < 1.$$

As such, our series is bounded by

$$\sum_{n=0}^{\infty} |a_n||z|^n < \sum_{n=0}^{\infty} r^n.$$

Since we know $r \in (0,1)$, this is a convergent geometric series, and as such the original series must also converge.

Thus, we only need to connect $\limsup_{n\to\infty} \sqrt[n]{|n|}$ with $\sqrt[n]{|a_n|}$.

It turns out that for any $\varepsilon > 0$, $\limsup b_n + \varepsilon$ is an upper bound for $\{a_n\}_{n\ge N}$.

We claim that there is some $\varepsilon > 0$ such that

$$|z|\left(\limsup_{n\to\infty} \sqrt[n]{|a_n|} + \varepsilon\right) < 1.$$

As such, there is some $N > 0$ such that for any $n > N$, $\sqrt[n]{|a_n|} < \limsup_{n\to\infty} \sqrt[n]{|a_n|} + \varepsilon$. This further means that $|z| \sqrt[n]{|a_n|} < 1$ for all $n > N$.

Repeating the same reasoning as before, we conclude that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

Next, suppose we have $|z| > R$. This means that

$$|z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1.$$

We claim that there is a subsequence indexed by n_k such that $|z| \cdot \sqrt[n_k]{\left|a_{n_k}\right|} > 1$.

Taking the n_kth power to both sides, we have $\left|a_{n_k}\right| |z|^{n_k} > 1$. Further, since this subsequence has all terms greater than 1, then $\lim_{k\to\infty} a_{n_k} z^{n_k} \neq 0$, which implies $\lim_{n\to\infty} |a_n||z|^n \neq 0$ as well.

This means that $\sum_{n=0}^{\infty} a_n z^n$ is divergent. □

This theorem tells us that any power series has a disk of radius R centered at the origin in which the series converges absolutely, and for any point outside of the disk, the series is divergent.

The natural question is then: what happens on the boundary of the disk? There is no guarantees for the behavior of power series along the boundary—we must consider it on a case-by-case basis.

Example 6.7

Let us consider the geometric series again. We've shown that for any $|z| = 1$, the series is divergent. (Briefly, because $|z| = 1$, then z^n cannot converge to zero, and as such the series is divergent.)

Example 6.8

Let us consider the series

$$\sum_{n=0}^{\infty} n z^n.$$

The convergence radius is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{n}} \to 1.$$

This is because $\limsup_{n\to\infty} \sqrt[n]{n} = 1$. As such, we have a unit disk in which the series converges absolutely within the disk, and diverges outside of it.

On the boundary, we have $|z| = 1$, in which case the series becomes

$$\sum_{n=0}^{\infty} n z^n = \sum_{n=0}^{\infty} n.$$

This series is divergent, as the limit of the terms is not zero.

Therefore, the original series diverges along the boundary.

Example 6.9

Let us consider the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

The convergence radius is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^2}}} = 1.$$

This is because $\limsup_{n\to\infty} \sqrt[n]{n^2} = 1$.

On the boundary, we have $|z| = 1$, in which case the series becomes

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This series converges by the p-series test, and as such the original series converges along the boundary.

Example 6.10

let us consider the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

The convergence radius is again $R = 1$, similar to before.

On the boundary, we have $|z| = 1$, in which case the series becomes

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series, and as such diverges. However, the original series may not be divergent (but it definitely does not converge absolutely).

Consider the case where $z = 1$; this is exactly the harmonic series, and as such the series diverges. Considering $|z| = 1$ but $z \neq 1$, the series actually converges—it converges conditionally.

To prove this, we will show a more general case. Suppose we have a complex sequence $\{a_n\}$, and let $A_n = \sum_{k=1}^{n} a_k$, and suppose $\{A_n\}$ is bounded (that is, there exists an $M > 0$ such that $|A_n| < M$ for any n).

Suppose we have another positive real sequence $\{b_n\}$ that decreases to zero (i.e. in our example, $b_n = \frac{1}{n}$).

We claim that $\sum_{n=1}^{\infty} a_n b_n$ is convergent. To prove this, we can show that the partial sums form a Cauchy sequence.

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} (A_k - A_{k-1}) b_k = \sum_{k=m}^{n} (A_k b_k - A_{k-1} b_k).$$

Writing this term by term, we have

$$A_n b_n - A_{n-1} b_n + A_{n-1} b_{n-1} - A_{n-2} b_{n-1} = \cdots + A_m b_m - A_{m-1} b_m.$$

We can then reorganize the sequence to group the A_i's together:

$$(A_n b_n - A_{m-1} b_m) + A_{n-1}(b_{n-1} - b_n) + A_{n-2}(b_{n-2} - b_{n-1}) + \cdots + A_m(b_m - b_{m+1}).$$

We want to show that this partial sum is convergent, so we want to give a bound for the norm:

$$\begin{aligned}\left|\sum_{k=m}^{n} a_k b_k\right| &\le |A_n b_n| + |A_{m-1} b_m| + \sum_{k=m}^{n-1} |A_k (b_k - b_{k+1})| \\ &\le |A_n b_n| + |A_{m-1} b_m| + \sum_{k=m}^{n-1} |A_k||b_k - b_{k+1}| \\ &\le |A_n b_n| + |A_{m-1} b_m| + \sum_{k=m}^{n-1} M(b_k - b_{k+1})\end{aligned}$$

Here, we used the boundedness of A_k and the decreasing nature of b_k to rewrite the last summation. If we write out all the terms, we have a telescoping sum:

$$\sum_{k=m}^{n-1} M(b_{k+1} - b_k) = M(b_{n-1} - b_n + b_{n-2} - b_{n-1} + \cdots + b_m - b_{m+1}) = M(b_m - b_n).$$

We can apply this to our case; we have $a_n = z_n$, with $|z| = 1$ but $z \neq 1$, and we have $b_n = \frac{1}{n}$. We can see that b_n decreases to zero, so we'd just need to show that A_n is bounded. We have from the formula for the partial sum of a geometric series as

$$A_n = z\frac{1 - z^n}{1 - z},$$

which we know is defined because $z \neq 1$. As such, let us consider the norm

$$|A_n| = |z|\frac{|1 - z^n|}{|1 - z|} \le \frac{1 + |z|^n}{|1 - z|} = \frac{2}{|1 - z|} = M_z.$$

This means that A_n is bounded, and the proposition holds.

2/8/2022

Lecture 7

Differentiating Power Series

Last time, we established that all points $z \in D_R(0)$ will make $\sum a_n z^n$ absolutely convergent, where R is the radius of convergence. The next natural question is: is this function holomorphic?

The most naive way to find the derivative of the infinite series $\sum a_n z^n$ is to differentiate term by term, since that's what we do for polynomials.

If we do this, then we have

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)' = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Let's first look at some properties of this RHS power series. We can first do a simple substitution $n = m + 1$ to give us

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} z^m.$$

Now that this is in the standard form of a power series, we can compute the convergence radius:

$$\hat{R} = \frac{1}{\limsup_{m\to\infty} \sqrt[m]{|m+1|a_{m+1}}}.$$

Looking just at the denominator, we have

$$\sqrt[m]{m+1} = (m+1)^{1/m} = (m+1)^{\frac{m+1}{m+1}\cdot\frac{1}{m}} \to 1^1.$$

This last equality is because $(m+1)^{\frac{1}{m+1}} \to 1$, and $\frac{m+1}{m} \to 1$ as well.

We also have

$$\sqrt[m]{|a_{m+1}|} = \sqrt[m+1]{|a_{m+1}|}^{\frac{m+1}{m}} \to 1/R.$$

As such, we know that $\hat{R} = R$.

This means that the derivative $\sum n a_n z^{n-1}$ has the same convergence radius as the original power series $\sum a_n z^n$.

Theorem 7.1: Differentiating a power series

$\sum a_n z^n$ is holomorphic on $D_R(0)$ and $(\sum a_n z^n)' = \sum n a_n z^{n-1}$.

Proof. Suppose we consider the power sum up to N terms:

$$f(z) = \sum a_n z^n = \sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n = S_N + E_N.$$

Here, S_N and E_N are defined respectively as the partial sum and the error term.

We can do the same thing with the derivative:

$$g(z) = \sum n a_n z^{n-1} = \sum_{n=0}^{\infty} n a_n z^{n-1} + \sum_{n=N+1}^{\infty} n a_n z^{n-1} = \hat{S}_N + \hat{E}_N.$$

We want to show that $S_N + E_N$ is holomorphic, and that $(S_N + E_N)' = \hat{S}_N + \hat{E}_N$. Since derivatives are linear, we can rewrite the LHS as $S_N' + E_N'$.

S_N contains finitely many terms, so it is a polynomial, and as such we already know that $S_N' = \hat{S}_N$. The only part left is to prove that $E_N' = \hat{E}_N$.

Suppose we take any $z_0 \in D_R(0)$. We want to show that $f'(z_0)$ exists, and $f'(z_0) = g(z_0)$.

Since $D_R(0)$ is an open set, there exists an $r > 0$ such that $D_r(z_0) \subseteq D_R(0)$. Looking at the definition of the derivative, we want to show

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = 0$$

We know that $f(z) = S_N(z) = E_N(z)$ and $g(z) = \hat{S}_N(z) + \hat{E}_N(z)$ for any N; splitting the above definition into two parts, we have

$$\begin{aligned}\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) &= \frac{S_N(z_0+h) - S_N(z_0)}{h} - \hat{S}_N(z_0) + \frac{E_N(z_0+h) - E_N(z_0)}{h} - \hat{E}_N(z_0) \\ &\le \underbrace{\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - \hat{S}_N(z_0)\right|}_{\to 0 \text{ as } h\to 0} + \left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| - \underbrace{\left|\hat{E}_N(z_0)\right|}_{\to 0 \text{ as } N\to\infty}\end{aligned}$$

The only term left goes to $\frac{0}{0}$ as $N \to \infty$ and $h \to 0$, so it's slightly trickier to compute.

Let us look at each term of the power series individually. We have

$$\frac{a_n(z_0+h)^n - a_n z_0^n}{h} = \frac{1}{h} a_n\big((z_0+h)^n - z_0^n\big).$$

We can utilize the fact that

$$a^n - b^n = (a-b)\big(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\big),$$

with $a = z_0 + h$ and $b = z_0$. The first term $(a-b)$ turns out to be h, which cancels out with the denominator, and we're left with

$$\frac{a_n((z_0+h_0)^n - z_0^n}{h} = a_n\big((z_0+h)^{n-1} + (z_0+h)^{n-2}z_0 + \cdots + z_0^{n-1}\big).$$

Taking the norm, we can use the triangle inequality to upper bound with

$$|a_n|\big(|z_0+h|^{n-1} + |z_0+h|^{n-2}|z_0| + \cdots + |z_0|^{n-1}\big).$$

Since we originally took z_0 and z_0+h to be within $D_R(0)$, both $|z_0|$ and $|z_0+h|$ can be bounded by $\hat{R}$ where $\hat{R} < R$. This means that we further have

$$\le |a_n|\big(\hat{R}^{n-1} + \hat{R}^{n-2}\hat{R} + \cdots + \hat{R}^{n-1}\big) \le |a_n| \cdot n\hat{R}^{n-1}.$$

This is an estimate of each individual term, so we can take the summation as

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \le \sum_{n=N+1}^{\infty} |a_n| n\hat{R}^{n-1}.$$

Since $\hat{R} < R$, this series converges (absolutely), since it is exactly the same form as the terms in $\sum a_n n z^{n-1}$, which we know to be convergent with convergence radius R.

Putting everything together, we can see that there exists an $N > 0$ such that

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \frac{\varepsilon}{3} \qquad \left|\hat{E}_N(z_0)\right| < \frac{\varepsilon}{3}$$

If we fix such an N, then we can further find some $h > 0$ small enough such that

$$\frac{S_N(z_0+h) - S_N(z_0)}{h} - \hat{S}_N(z_0) < \frac{\varepsilon}{3}.$$

This means that for any $\varepsilon > 0$, there exists an h (and an N) such that $\frac{f(z_0+h)-f(z_0)}{h} < \varepsilon$. Hence, the limit

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \to 0,$$

and f is holomorphic at z_0, with derivative $g(z_0)$. □

Example 7.2

Suppose we have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We've established that the radius of convergence is $+\infty$, and as such e^z is holomorphic on $\mathbb{C}$ (an entire function). Further, we can calculate the derivative:

$$\left(e^z\right)' = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{z^m}{m!},$$

where $m = n-1$. Since this is exactly the same exponential function power series, e^z is its own derivative.

Example 7.3

Suppose we consider
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$
We can see the convergence radius is $+\infty$, since $\{a_n\} = 1, -\frac{1}{3!}, \frac{1}{5!}, \dots$ and its nth root has a supremum of zero.

Similarly, we have
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)1}.$$
The convergence radius is again $+\infty$.

This means that both $\sin z$ and $\cos z$ are entire functions. Looking at the derivative of $\sin z$, we can differentiate term by term:
$$(\sin z)' = 1 - \frac{3z^2}{3!} + \frac{5z^4}{4!} - \frac{7z^6}{7!} + \cdots = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \cos z.$$
The same applies to $\cos z$, giving us $(\cos z)' = -\sin z$.

Example 7.4: Euler's Formula

Suppose we look at e^{iz}:
$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}.$$
We know that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$, and as such we can split this summation up into two parts; terms with i and terms without i. Doing this, we have
$$e^{iz} = \cos z + i \sin z.$$
This is Euler's formula.

One remark: for $z \in \mathbb{R}$, we know that $|\sin z| \le 1$ and $|\cos z| \le 1$. However, for $z \in \mathbb{C}$, this is not true. To see this, suppose we have $z = x + iy$. Plugging into e^{iz}, we have
$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}e^{ix}.$$
Looking at the norm, we have
$$\left|e^{iz}\right| = \left|e^{-y}\right|\left|e^{ix}\right| = e^{-y}.$$
We have $\left|e^{ix}\right| = 1$ since it is on the unit circle, so all we have left is e^{-y}. For negative y, we can get arbitrarily large values, and as such e^{iz} is not bounded in general.

Further, if we plug in e^{-iz}, we have
$$e^{-iz} = \cos(-z) + i\sin(-z) = \cos z - i \sin z.$$
This means that
$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$
$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
The unboundedness of e^{iz} means that $\sin z$ and $\cos z$ are both unbounded on $\mathbb{C}$.

2/10/2022

Lecture 8

Integration Along Curves

Before talking about integration, let us first talk about curves.

With a curve, we have $z : [a,b] \to \mathbb{C}$ (for $a, b \in \mathbb{R}$); it maps $t \mapsto z(t)$, where t denotes time. In other words, we go from point $z(a)$ to point $z(b)$ through time. This can be rewritten as $z(t) = x(t) + i\,y(t)$ for $x(t), y(t) \in \mathbb{R}$.

We will specifically be working with *smooth curves*.

Definition 8.1: Smooth Curve

For a curve $z : [a,b] \to \mathbb{C}$, we say that it is smooth if $z'(t) = x'(t) + i\,y'(t)$ exists for every $t \in [a,b]$, and that z' is continuous. Further, we require $z'(t) \neq 0$ for any $t \in [a,b]$.

Definition 8.2: Piecewisely Smooth Curve

A curve $z : [a,b] \to \mathbb{C}$ is *piecewise smooth* if there is a finite partition $\{a = a_0, a_1, \ldots, a_{n-1}, a_n = b\}$ such that all $z_i : [a_{i-1}, a_i] \to \mathbb{C}$ are smooth.

Definition 8.3: Simple Curve

A curve $z : [a,b] \to \mathbb{C}$ is *simple* if it does not intersect itself. Formally, z is simple if there does not exist a pair $t_1 \neq t_2$ such that $z(t_1) = z(t_2)$. In another words, z is simple if $z : (a,b) \to \mathbb{C}$ is injective.

Note that we exclude the endpoints here, because later we want to consider loops, in which case $z(a) = z(b)$.

Definition 8.4: Loop

A curve $z : [a,b] \to \mathbb{C}$ is a loop if $z(a) = z(b)$.

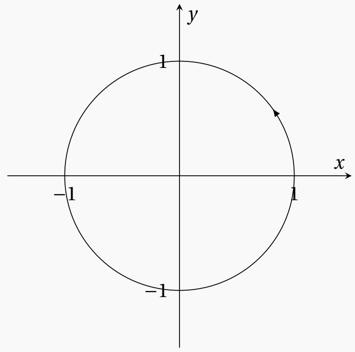
Definition 8.5: Simple Loop

A curve $z : [a,b] \to \mathbb{C}$ is a *simple loop* if it is a simple curve and it is a loop.

That is, $z(a) = z(b)$ and $z : (a,b) \to \mathbb{C}$ is injective.

Example 8.6

Suppose we consider a circle of radius R in $\mathbb{C}$.



There are many ways to parameterize this curve if we start at $z = 1$. A first way is to define $z : [0, 2\pi] \to \mathbb{C}$ as

$$z(t) = e^{it} = \cos t + i \sin t.$$

We can verify that $\cos t$ and $\sin t$ are both differentiable, and their derivatives are never zero at the same time.

Another way to parameterize the curve is $z : [0, \pi] \to \mathbb{C}$ such that

$$\tilde{z}(t) = e^{2it}.$$

This parameterization is just "double speed".

Definition 8.7: Equivalent Parameterizations

We consider two parameterizations of smooth curves $z_1 : [a_1, b_1] \to \mathbb{C}$ and $z_2 : [a_2, b_2] \to \mathbb{C}$ to be *equivalent* if we can find some map $\phi : [a_1, b_1] \to [a_2, b_2]$ with $\phi \in C^1$ and $\phi'(t) > 0$ for all $t \in [a_1, b_1]$ such that $z_1 = z_2 \circ \phi$.

That is, $z_1(t) = z_2(\phi(t))$ for all $t \in [a_1, b_1]$.

Note that we require a positive derivative, because we do not want to change the orientation of the curve.

Example 8.8

In the earlier example, with $z_1 : [0, 2\pi] \to \mathbb{C}$ defined as $z_1(t) = e^{it}$ and $z_2 : [0, \pi] \to \mathbb{C}$ defined as $z_2(t) = e^{2it}$, we have

$$\phi : [0, 2\pi] \to [0, \pi], \ \phi(t) = \frac{1}{2} t.$$

This map is a smooth map, and its derivative $\phi'(t) = \frac{1}{2} > 0$, as required.

Example 8.9

Another way to parameterize the unit circle in $\mathbb{C}$ is with

$$z_3 : [0, 2\pi] \to \mathbb{C}, \ z_3(t) = e^{-it}.$$

In contrast to z_1 and z_2, which go counterclockwise around the circle, z_3 goes clockwise around the circle.

If we try to find some map $\phi : [0, 2\pi] \to [0, 2\pi]$, such as $\phi = -t$, to find equivalence between z_1 and z_3, we'd find that $\phi' < 0$. This means that z_1 and z_3 are not equivalent parameterizations.

As a remark, we can show that this equivalence between parameterizations is an equivalence relation:

- Reflexive: $z \sim z$, with $\phi(t) = t$
- Symmetric: $z_1 \sim z_2 \implies z_2 \sim z_1$, with ϕ^{-1} (if $z_1 \sim z_2$ through ϕ)
- Transitive: $z_1 \overset{\phi_1}{\sim} z_2 \wedge z_2 \overset{\phi_2}{\sim} z_3 \implies z_1 \sim z_3$ through $\phi_2 \circ \phi_1$

Definition 8.10: Length of Smooth Curve

Suppose we have a smooth curve $z : [a, b] \to \mathbb{C}$. We calculate the length

$$\ell(z) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t.$$

Here, we define $z'(t) = x'(t) + i y'(t)$, which means that $\left|z'(t)\right| = \sqrt{x'(t)^2 + y'(t)^2}$; this means that alternatively,

$$\ell(z) = \int_a^b \left|z'(t)\right| \mathrm{d}t.$$

It turns out that this definition of length does not depend on the parameterization, which is desired. Specifically, if we have $z_1 : [a_1, b1] \to \mathbb{C}$ and $z_2 : [a_2, b_2] \to \mathbb{C}$ are similar ($z_1 \sim z_2$ through ϕ), then $\ell(z_1) = \ell(z_2)$.

To see this, we have $z_2 \circ \phi = z_1$, which means

$$z_1'(t) = z_2'(\phi(t)) \cdot \phi'(t).$$

Taking the norm, we have $\left|z_1'(t)\right| = \left|z_2'(\phi(t))\right|\left|\phi'(t)\right|$, and $\left|\phi'(t)\right| = \phi'(t)$ since it is positive. Plugging this in,

$$\begin{aligned} \ell(z_1) &= \int_{a_1}^{b_1} \left|z_1'(t)\right| \mathrm{d}t \\ &= \int_{a_1}^{b_1} \left|z_2'(\phi(t))\right| \phi'(t) \, \mathrm{d}t \end{aligned}$$

Substituting $s = \phi(t)$, with $\mathrm{d}s = \phi'(t) \, \mathrm{d}t$, we have

$$= \int_{a_2}^{b_2} \left|z_2'(s)\right| \mathrm{d}s$$

For notation, suppose we denote z^- as the parameterization with the reverse orientation. Specifically, we have

$$z^-(t) = z(a + b - t).$$

It turns out that the length should still be the same if we take z^- instead of z (which should also be expected). This is because we have an extra negative sign by removing the norm in $\phi'(t)$, but at the same time, we swap the limits of integration, which cancels out the negative sign.

Suppose we consider a $\gamma \subseteq \mathbb{C}$ as a curve with a smooth parameterization $z : [a, b] \to \mathbb{C}$. We say f as a map from the image of γ to $\mathbb{C}$ is continuous defined over γ, if there is a smooth parameterization (and thus all parameterizations are smooth) z so that $f \circ z : [a, b] \to \mathbb{C}$ is continuous.

Definition 8.11: Contour Integral

For a continuous map f over γ, we define

$$\int_\gamma f(z) \, \mathrm{d}z := \int_a^b f(z(t)) \cdot z'(t) \, \mathrm{d}t,$$

for some parameterization $z : [a, b] \to \mathbb{C}$ of γ. (The multiplication here is complex.)

Example 8.12

Suppose we want to calculate

$$\int_\gamma z\,\mathrm{d}z,$$

for γ as the circle with radius R. Suppose we take a parameterization of γ as $z\colon [0,2\pi] \to \mathbb{C}$ such that $z(t) = Re^{it}$. This means we have

$$\int_\gamma z\,\mathrm{d}z = \int_0^{2\pi} z(t)\cdot z'(t)\,\mathrm{d}t$$

We can see that $z'(t) = Rie^{it}$, and as such we have

$$\begin{aligned}
&= \int_0^{2\pi} Re^{it}\cdot Rie^{it}\,\mathrm{d}t \\
&= R^2 i \int_0^{2\pi} e^{2it}\,\mathrm{d}t \\
&= R^2 i \int_0^{2\pi} \cos(2t) + i\sin(2t)\,\mathrm{d}t \\
&= \int_0^{2\pi} -R^2\sin(2t) + i(R^2\cos(2t))\,\mathrm{d}t && \text{(Euler's formula)} \\
&= \int_0^{2\pi} -R^2\sin(2t)\,\mathrm{d}t + i\int_0^{2\pi} R^2\cos(2t)\,\mathrm{d}t && \text{(splitting real and imaginary)} \\
&= 0 + 0 = 0
\end{aligned}$$

Here, the integrals evaluate to zero, since integrating over a period of $\sin(2t)$ cancels out all areas, and the same goes for $\cos(2t)$; integrating over a period cancels out all areas.

It actually turns out that integrating any entire function over any circle with any radius and center will always give us zero (something to show later).

Example 8.13

Suppose we have $f(z) = \frac{1}{z}$ for $z \neq 0$. Taking $z\colon [0,2\pi] \to \mathbb{C}$ as a parameterization of a circle γ, suppose we want to calculate

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{z(t)} z'(t)\,\mathrm{d}t$$

We have $z'(t) = Rie^{it}$, so plugging this in, we have

$$\begin{aligned}
&= \int_0^{2\pi} \frac{Rie^{it}}{Re^{it}}\,\mathrm{d}t \\
&= \int_0^{2\pi} i\,\mathrm{d}t \\
&= i\int_0^{2\pi} \mathrm{d}t \\
&= 2\pi i
\end{aligned}$$

Theorem 8.14: Contour integrals are independent of parameterization

The definition

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t$$

is independent of the choice of equivalent parameterizations of γ.

Proof. Suppose we parameterize γ with $z_1 : [a_1, b_1] \to C$ and $z_2 : [a_2, b_2] \to \mathbb{C}$ such that $z_1 \sim z_2$ through $\phi : [a_1, b_1] \to [a_2, b_2]$. That is, $z_1 = z_2 \circ \phi$.

We then have two integrals we want to prove equality:

$$\int_{a_1}^{b_1} f(z_1(t))z_1'(t)\,\mathrm{d}t$$
$$\int_{a_2}^{b_2} f(z_2(s))z_2'(s)\,\mathrm{d}s$$

We have $z_1'(t) = z_2'(\phi(t_1)) \cdot \phi'(t)$, so plugging this in, we have

$$\int_{a_1}^{b_1} f(z_1(t))z_1'(t)\,\mathrm{d}t = \int_{a_1}^{b_1} f(z_2(\phi(t))) \cdot z_2'(\phi(t)) \cdot \phi'(t)\,\mathrm{d}t$$

Taking $s = \phi(t)$ with $\mathrm{d}s = \phi'(t)\,\mathrm{d}t$, we have

$$= \int_{a_2}^{b_2} f(z_2(s))z_2'(s)\,\mathrm{d}s$$

This is exactly the same integral for the parameterization z_2, and as such integrating along a curve is independent of the parameterization of the curve. □

As a remark, if we reverse the orientation of the parameterization, the integral will flip signs, since we no longer have the norm.

2/15/2022

Lecture 9

Properties of Contour Integration, Primitives

9.1 Properties of Contour Integration

Here are some properties of complex integration of a continuous function along smooth curves.

Lemma 9.1: Linearity of contour integrals

Suppose γ is a smooth curve, and f and g are two continuous functions over γ. For any $\alpha, \beta \in \mathbb{C}$, we have

$$\int_\gamma \big(\alpha f(z) + \beta g(z)\big)\,\mathrm{d}z = \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z)\,\mathrm{d}z.$$

Proof. Suppose we have a smooth parameterization $z : [a, b] \to \mathbb{C}$ for γ. We have

$$\int_\gamma \big(\alpha f(z) + \beta g(z)\big)\,\mathrm{d}z = \int_a^b \big(\alpha f(z(t)) + \beta g(z(t))\big) z'(t)\,\mathrm{d}t$$

$$
\begin{aligned}
&= \alpha \int_a^b f(z(t))z'(t)\,\mathrm{d}t + \beta \int_a^b g(z(t))z'(t)\,\mathrm{d}t \\
&= \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z)\,\mathrm{d}z
\end{aligned}
$$

□

Lemma 9.2: Contour integral with reverse orientation

Suppose γ is a smooth curve, and f is a continuous function over γ. If γ^- is the curve with reverse orientation, we have

$$\int_{\gamma^-} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

Proof. Suppose we have a smooth parameterization $z : [a,b] \to \mathbb{C}$ for γ. We can also define $z^-(t) := z(a+b-t)$ as a smooth parameterization of γ^-. We then have

$$
\begin{aligned}
\int_{\gamma^-} f(z)\,\mathrm{d}z &= \int_a^b f(z^-(t)){z^-}'(t)\,\mathrm{d}t \\
&= \int_a^b f(z(a+b-t))z'(a+b-t)(-1)\,\mathrm{d}t
\end{aligned}
$$

Letting $s = a+b-t$, we have $\mathrm{d}s = -1$; also substituting the integral bounds, we have

$$
\begin{aligned}
&= \int_b^a f(z(s))z'(s)\,\mathrm{d}s \\
&= -\int_a^b f(z(s))z'(s)\,\mathrm{d}s \\
&= -\int_\gamma f(z)\,\mathrm{d}z
\end{aligned}
$$

□

Lemma 9.3: Estimation Lemma

Suppose γ is a smooth curve, and f is a continuous function over γ. We have

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \sup_{z\in\gamma}\left|f(z)\right| \cdot \mathrm{length}(\gamma).$$

Proof. Suppose we have a smooth parameterization $z : [a,b] \to \mathbb{C}$. We have

$$
\begin{aligned}
\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_a^b f(z(t))z'(t)\,\mathrm{d}t\right| \\
&\le \int_a^b \left|f(z(t))z'(t)\right|\mathrm{d}t \\
&= \int_a^b \left|f(z(t))\right|\left|z'(t)\right|\mathrm{d}t
\end{aligned}
$$

Here we can bound $\left|f(z(t))\right|$ with its largest attained value, which is a constant and can be pulled out of the integral:

$$\begin{aligned}
&\leq \sup_{z\in\gamma}\left|f(z)\right| \cdot \int_a^b \left|z'(t)\right| \mathrm{d}t \\
&= \sup_{z\in\gamma}\left|f(z)\right| \cdot \operatorname{length}(\gamma)
\end{aligned}$$

□

Note that integration of a continuous function over a *piecewise* smooth curve is defined as the sum of the integrals over each piece of the curve. It should be clear (from previous results) that the integral is independent of choices of the parameterizations of each piece, and also independent of our choice of divisions of the curve.

Corollary 9.4

Suppose γ is a *piecewise* smooth curve, and f and g are continuous functions over γ. Then, all three previous results in Lemmas 9.1 to 9.3 still hold.

9.2 Primitives

Definition 9.5: Primitive

Suppose f is a function over an open set $\Omega \subseteq \mathbb{C}$. A *primitive* of f is a holomorphic function F over Ω whose derivative is f. That is, $F'(z) = f(z)$.

Example 9.6

Suppose we have the function $f(z) = z$.

A primitive of f is the function $F(z) = \frac{1}{2}z^2$ defined over $\mathbb{C}$. Further, any function $\frac{1}{2}z^2 + c$ is also a primitive, for $c \in \mathbb{C}$.

Example 9.7

We'll see later that the function $f(z) = \frac{1}{z}$ has no primitive over $\mathbb{C}*$ (i.e. $\mathbb{C} \setminus \{0\}$.

Theorem 9.8: Fundamental Theorem of Calculus for Contour Integrals

Suppose we have a continuous function f with primitive F over the open set $\Omega \subseteq \mathbb{C}$, and further suppose γ is a piecewise smooth curve on Ω starting at z_0 and ending at z_1. We then have

$$\int_\gamma f(z)\,\mathrm{d}z = F(z_1) - F(z_0).$$

In particular, when γ is a loop, $\oint_\gamma f(z)\,\mathrm{d}z = 0$.

Proof. As a first case, suppose γ is a smooth curve with parameterization $z : [a,b] \to \mathbb{C}$. We have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b F'(z(t))z'(t)\,\mathrm{d}t$$

$$= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(z(t))\,\mathrm{d}t$$

Crucially here, we make use of the fundamental theorem of calculus for real functions:

$$\begin{aligned} &= F(z(b)) - F(z(a)) \\ &= F(a_1) - F(z_0) \end{aligned}$$

To generalize to piecewise smooth curves, suppose γ is piecewise smooth with division

$$a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$$

over $[a, b]$, such that there exists a parameterization $z : [a, b] \to \mathbb{C}$ which is continuous on $[a, b]$ and smooth on each piece.

Suppose we define γ_i to be the curve over $[a_i, a_{i+1}]$, for $i = 0, 1, \ldots, n-1$. We can then use the smooth case proved earlier to show that

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \sum_{i=0}^{n-1} \int_{\gamma_i} f(z)\,\mathrm{d}z \\ &= \sum_{i=0}^{n-1} \left(F(z(a_{i+1})) - F(z(a_i))\right) \end{aligned}$$

Here, the terms in the series telescope, leaving just the first and last:

$$\begin{aligned} &= F(z(b)) - F(z(a)) \\ &= F(z_1) - F(z_0) \end{aligned}$$

□

Example 9.9

For any polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with $a_n \neq 0$, it has a primitive

$$F(z) = a_0 z + \frac{1}{2} a_1 z^2 + \cdots + \frac{1}{n+1} a_n z^{n+1}.$$

However, primitives of f aren't unique; we can add any constant $c \in \mathbb{C}$, and the function $F(z) + c$ is also a primitive.

We'll show later that all primitives of f take on this form.

Example 9.10

We can also see from Theorem 9.8 that $f(z) = \frac{1}{z}$ has no primitive on $\Omega = \mathbb{C} \setminus \{0\}$.

This is because we've found earlier that

$$\oint_{C_1^+} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0,$$

for C_1^+ as the counterclockwise oriented disk of radius 1; we'd expect this integral to be zero from Theorem 9.8 if f has a primitive, since we're integrating over a closed loop.

Lemma 9.11

Suppose $\Omega \subseteq \mathbb{C}$ is a connected open set in $\mathbb{C}$, and $F' = 0$. Then, F must be constant over Ω.

Proof. Suppose we have $z_0 \in \Omega$, and we consider the set

$$A := \{z \in \Omega \mid F(z) = F(z_0)\}.$$

Note that A is never an empty set, since $z_0 \in A$ (i.e. $F(z_0) = F(z_0)$ always).

Making use of the fact that the preimage of a continuous function on a closed set is also closed, we can observe that A is the preimage of the closed set $\{F(z_0)\}$ containing one point.

We can also see that A is open in Ω. In particular, we need to show that any $w_0 \in A$ is an interior point of A.

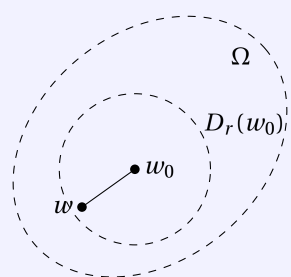
Firstly, we know that Ω is open, so there exists some $r > 0$ such that $D_r(w_0) \subset \Omega$ for our prior $w_0 \in A$. Further, consider any $w \in D_r(w_0)$. We want to show that $F(w) = F(w_0) = F(z_0)$, and as such $w \in A$ as well (making the arbitrary $w \in D_r(w_0)$ an interior point of A); to do this, consider the line segment between w_0 and w defined by

$$z : [0,1] \to \mathbb{C}, \quad z(t) = (1-t)w_0 + tw.$$

Since z is smooth for $t \in [0,1]$, we then have

$$|z(t) - w_0| = |(1-t)w_0 + tw - w_0| = t|w - w_0| < tr \leq r.$$

This shows that the line segment lies entirely within the disk $D_r(w_0)$; this can also be seen easily geometrically:



As such, we can now use Theorem 9.8 to show that

$$F(w) - F(w_0) = \int_L F'(z)\,\mathrm{d}z = \int_L 0\,\mathrm{d}z = 0,$$

This proves that any $w \in D_r(w_0)$ is also an interior point to A, so A is an open set.

Since we've assumed that Ω is connected, we can utilize the fact that any nonempty open and closed subset in Ω can only be Ω itself to see that $A = \Omega$.

Equivalently, for any $z \in \Omega$, we must have $F(z) = F(z_0)$, i.e. a constant. □

We can now use Lemma 9.11 to show that all primitives of a given function f on the open region $\Omega \subset \mathbb{C}$ must only differ by a constant term. Specifically, suppose F_1 and F_2 are both primitives of f. We have

$$(F_1 - F_2)' = F_1' - F_2' = f - f = 0,$$

showing that $(F_1 - F_2)' = 0$, and by Lemma 9.11, $F_1 - F_2$ must be constant on Ω, i.e. differing by at most a constant term.

2/17/2022

Lecture 10

Integration over Closed Loops

In the past few lectures, we've looked at complex integrals, and especially the integration over the unit circle with the origin at its center. Specifically, we've found that for the circle γ,

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0.$$

We want to eventually prove the fact that for any holomorphic $f(z)$ on an open set Ω, the integral

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0,$$

for a closed loop γ.

Consider a curve γ, which encloses a region D, all in the open set Ω. We want to look at $\oint_{\gamma} f(z)\,\mathrm{d}z$.

Here, we have $f(z) = u(z) + i\,v(z)$. When computing the integral, we'd want

$$z'(t) = (x(t) + i\,y(t))' = x'(t) + i\,y'(t).$$

The integrand would be

$$f(z(t)) \cdot z'(t) = (u(z(t)) + i\,v(z(t))) \cdot \left(x'(t) + i\,y'(t)\right)$$

We can split this into two parts:

$$= \left(u(z(t)) \cdot x'(t) - v(z(t)) \cdot y'(t)\right) + i\left(v(z(t)) \cdot x'(t) + u(z(t)) \cdot y'(t)\right)$$

Calculating the complex integral, we have

$$\oint_{\gamma} f(z)\,\mathrm{d}z = \int_a^b u(z(t))x'(t) - v(z(t))y'(t)\,\mathrm{d}t + i\int_a^b v(z(t))x'(t) + u(z(t))y'(t)\,\mathrm{d}t$$

Taking $x'(t)\,\mathrm{d}t$ as $\mathrm{d}x$, and $y'(t)\,\mathrm{d}t$ as $\mathrm{d}y$, we can rewrite this as two path integrals:

$$= \oint_{\gamma} u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y + i\oint_{\gamma} v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y$$

If the partial derivatives of u and v are both continuous, then we can use Green's formula, rewriting this as a double integral over the region that γ encloses:

$$\oint_{\gamma=\partial D} u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y = -\iint_D \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y$$
$$\oint_{\gamma=\partial D} v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y = -\iint_D \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y$$

Recall the Cauchy-Riemann equations; we know that since f is holomorphic (which is what we're assuming) the partial derivatives of u and v satisfy

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}.$$

This means that we can simplify

$$-\iint_D \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y = -\iint_D 0\,\mathrm{d}x\,\mathrm{d}y = 0$$
$$-\iint_D \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y = -\iint_D 0\,\mathrm{d}x\,\mathrm{d}y = 0$$

This means the original integral is always zero.

However, we don't actually know whether $u, v \in C^1$; we don't know whether the partial derivatives of u and v are continuous (they actually are). As such, we'd like to find another way to prove this result (and it turns out that this can be used to show that u, v are indeed in C^1).

Theorem 10.1: Cauchy's Theorem (over a disk)

Let D is an open disk in $\mathbb{C}$, and f is a holomorphic function over D. Then, for any piecewisely smooth curve γ inside D, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. The idea here is to try to prove that f has a primitive on D; from last time, we've already shown that any function that has a primitive has a complex integral evaluating to zero over a loop.

As such, we claim that any holomorphic function on a disk has primitives. □

Lemma 10.2

Any holomorphic function on a disk has primitives.

Proof. We will use Goursat's theorem (Theorem 10.3) to prove this result. We want to define a holomorphic F on D such that $F'(z) = f(z)$.

Suppose we look at a point z_0, and any other point z in D; we know that

$$F(z) - F(z_0) = \int_{\gamma_{z_0,z}} f(z)\,\mathrm{d}z \implies F(z) = \int_{\gamma_{z_0,z}} f(z)\,\mathrm{d}z + F(z_0).$$

As such, suppose z_0 is the center of the disk in consideration, and for any $z \in D$, we let $\gamma_{z_0,z}$ denote the path from z_0 to z. We then define

$$F(z) = \int_{\gamma_{z_0,z}} f(w)\,\mathrm{d}w,$$

where we know we have

$$F(z_0) = \int_{\gamma_{z_0,z_0}} f(w)\,\mathrm{d}w = 0.$$

We want to show that $F(z)$ defined on D is holomorphic, and that $F'(z) = f(z)$.

Suppose we take h, with $|h|$ small enough such that $z + h \in D$. We want to show that $\frac{F(z+h)-F(z)}{h} - f(z) \to 0$ as $h \to 0$. We have

$$F(z+h) - F(z) = \int_{\gamma_{z_0,z+h}} f(w)\,\mathrm{d}w - \int_{\gamma_{z_0,z}} f(w)\,\mathrm{d}w$$

Notice that we have a triangle here; by Goursat's theorem, the integration over the triangle is zero; writing this integral in terms of its three sides, we have

$$\int_{\gamma_{z_0,z+h}} f(w)\,\mathrm{d}w + \int_{\gamma_{z+h,z}} f(w)\,\mathrm{d}w + \int_{\gamma_{z,z_0}} f(w)\,\mathrm{d}w = 0$$

We know that $\gamma_{z,z_0} = \gamma^-_{z_0,z}$, and as such we have

$$\begin{aligned}
\int_{\gamma_{z_0,z+h}} f(w)\,\mathrm{d}w + \int_{\gamma_{z+h,z}} f(w)\,\mathrm{d}w - \int_{\gamma_{z_0,z}} f(w)\,\mathrm{d}w &= 0 \\
F(z+h) + \int_{\gamma_{z+h,z}} f(w)\,\mathrm{d}w - F(z) &= 0 \\
F(z+h) - F(z) = -\int_{\gamma_{z+h,z}} f(w)\,\mathrm{d}w &= \int_{\gamma_{z,z+h}} f(w)\,\mathrm{d}w
\end{aligned}$$

This means that we can rewrite our earlier complex derivative:

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{\gamma_{z,z+h}} f(w)\,\mathrm{d}w - f(z)$$

Noting that

$$\int_{\gamma_{z,z+h}} f(z)\,\mathrm{d}w = f(z)\int_{\gamma_{z,z+h}} \mathrm{d}w = hf(z),$$

we have

$$\begin{aligned}
\frac{F(z+h)-F(z)}{h} &= \frac{1}{h}\int_{\gamma_{z,z+h}} f(w)\,\mathrm{d}w - \frac{1}{h}\int_{\gamma_{z,z+h}} f(z)\,\mathrm{d}w \\
&= \frac{1}{h}\int_{\gamma_{z,z+h}} \big(f(w)-f(z)\big)\,\mathrm{d}w
\end{aligned}$$

We can estimate the norm of this expression; showing the norm converges to zero is equivalent to saying the original expression converges to zero as well.

$$\left|\frac{1}{h}\int_{\gamma_{z,z+h}} \big(f(w)-f(z)\big)\,\mathrm{d}w\right| = \frac{\left|\int_{\gamma_{z,z+h}} \big(f(w)-f(z)\big)\,\mathrm{d}w\right|}{|h|}$$

From the inequality from last time, this expression is bounded by

$$\le \frac{\sup_{w\in\gamma_{z,z+h}}\big|f(w)-f(z)\big|\cdot \operatorname{length}(\gamma_{z,z+h})}{|h|}$$

Since the length of the path is exactly $|h|$, this cancels out to give us

$$= \sup_{w\in\gamma_{z,z+h}} \big|f(w)-f(z)\big|$$

As $h\to 0$, the space of possible w's gets smaller and smaller, and the set of points gets closer and closer to z. Since f is continuous at z, this means that $\sup_{w\in\gamma_{z,z+h}}\big|f(w)-f(z)\big| \to 0$ as $h\to 0$. □

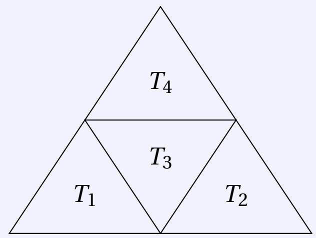
Theorem 10.3: Goursat's Theorem

Let Ω be an open set in $\mathbb{C}$ and T is a closed triangle in Ω (i.e. it includes its boundary), with f as a holomorphic function on Ω. Then, we have

$$\oint_{\partial T} f(z)\,\mathrm{d}z = 0.$$

(As a note, when talking about an integral over ∂T, we're referring to the natural orientation induced by T; that is, the counter-clockwise orientation, keeping the region on the left.)

Proof. Suppose we have the following triangle in an open set Ω, divided into four triangles (by drawing lines between their midpoints) as shown:



Observing that the inner small triangle directly cancels out integrals over its sides, we can write

$$\int_{\partial T} f(z)\,\mathrm{d}z = \int_{\partial T_1} f(z)\,\mathrm{d}z + \int_{\partial T_2} f(z)\,\mathrm{d}z + \int_{\partial T_3} f(z)\,\mathrm{d}z + \int_{\partial T_4} f(z)\,\mathrm{d}z.$$

We can bound the norm of the RHS summation by $4\int_{\partial T_{max}} f(z)\,\mathrm{d}z$, where T_{max} is the triangle with maximum integral.

Further, we can keep going, dividing this triangle into four, until we get closer and closer to one of the vertices.

Formally, suppose $T = T^0$. We've shown that if we divide this triangle into four, each denoted by T_i^1 for $1 \le i \le 4$, we have the inequality

$$\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \le \left|\int_{\partial T_1^1} f(z)\,\mathrm{d}z\right| + \left|\int_{\partial T_2^1} f(z)\,\mathrm{d}z\right| + \left|\int_{\partial T_3^1} f(z)\,\mathrm{d}z\right| + \left|\int_{\partial T_4^1} f(z)\,\mathrm{d}z\right|.$$

Suppose we choose $n_1 \in \{1,2,3,4\}$ such that $\left|\int_{\partial T_{n_1}^1 f(z)\,\mathrm{d}z}\right|$ is maximal among the four integrals. This means we have

$$\left|\int_{\partial T^0} f(z)\,\mathrm{d}z\right| \le 4 \cdot \left|\int_{\partial T_{n_1}^1 f(z)\,\mathrm{d}z}\right|.$$

Repeating the division to $\Delta T_{n_1}^1$, we get the four triangles $T_1^2, T_2^2, T_3^2, T_4^2$. This means we have

$$\left|\int_{\partial T^0} f(z)\,\mathrm{d}z\right| \le 4^2 \left|\int_{T_{n_2}^2} f(z)\,\mathrm{d}z\right|.$$

After k steps, we have the bound

$$\left|\int_{\partial T^0} f(z)\,\mathrm{d}z\right| \le 4^k \left|\int_{\partial T_{n_k}^k} f(z)\,\mathrm{d}z\right|.$$

Notice that each time we make a partition, the diameter of each triangle halves. This means that after k steps, the diameter of T^k is $\frac{1}{2^k}\mathrm{diam}(T^0)$. Further, we know that

$$T = T^0 \supseteq T_{n_1}^1 \supseteq T_{n_2}^2 \supseteq \cdots$$

is a sequence of nested compact sets.

By Theorem 3.3, there is a unique point $z \in \cap_{n=1}^{\infty} T_{n_k}^k \subseteq T$.

Because $f(z)$ is holomorphic at z_0, in a neighborhood of z_0, we can write $\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) = \psi(z) \to 0$ as $z \to z_0$. Rearranging, we have

$$f(z) = \big(f'(z_0) + \psi(z)\big)(z - z_0) + f(z_0)$$

$$\int_{\partial T_{n_k}^k} f(z)\,\mathrm{d}z = \int_{\partial T_{n_k}^k} f'(z_0)(z-z_0)\,\mathrm{d}z + \int_{\partial T_{n_k}^k} \psi(z)(z-z_0)\,\mathrm{d}z + \int_{\partial T_{n_k}^k} f(z_0)\,\mathrm{d}z$$

Looking at the last integral, since z_0 is fixed, we're integrating over a constant over a loop, and since any constant has a primitive, we know that the last integral evaluates to zero.

Looking at the first integral, since $f'(z_0)$ is again a constant, we can pull it out, and what's left is linear, which has a primitive $\frac{1}{2}z^2 - z_0 z$, and as such this also evaluates to zero.

Now, we only have the second integral left, which we can estimate its norm:

$$\begin{aligned}\left|\int_{\partial T^k_{n_k}} \psi(z)(z-z_0)\,\mathrm{d}z\right| &\le \sup\left|\psi(z)(z-z_0)\right| \cdot \operatorname{length}(\partial T^k_{n_k}) \\ &\le \sup\left|\psi(z)\right||z-z_0| \cdot \operatorname{length}(\partial T^k_{n_k}) \\ &\le \sup\left|\psi(z)\right| \cdot \sup|z-z_0| \cdot \operatorname{length}(\partial T^k_{n_k})\end{aligned}$$

We know the first two terms converges to zero as $z \to z_0$, and further the last term can also be bounded by some constant (here, C) times the diameter of the triangle, which we know also converges to zero:

$$\begin{aligned}&\le \sup\left|\psi(z)\right| \operatorname{diam}(T^k_{n_k}) \cdot \operatorname{diam}(T^k_{n_k}) \cdot C \\ &\le \sup\left|\psi(z)\right| \cdot \left(\frac{1}{2^k}\operatorname{diam}(T)\right)^2 \\ &= \sup\left|\psi(z)\right| \cdot \frac{1}{4^k}\operatorname{diam}(T)^2\end{aligned}$$

Notice that this 4^{-k} appears to exactly cancel out the decrease in size as we divide more triangles. This means that we have

$$\begin{aligned}\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| &\le 4^k \left|\int_{\partial T^k_{n_k}} f(z)\,\mathrm{d}z\right| \\ &\le 4^k \sup\left|\psi(z)\right| \cdot \frac{1}{4^k}\operatorname{diam}(T) \\ &= \sup\left|\psi(z)\right| \cdot C\end{aligned}$$

Further, as $k \to \infty$, $|z - z_0| \to 0$ since $z \in \partial T^k_{n_k}$, and as such $\left|\psi(z)\right| \to 0$. This shows that the integral over this triangle must be zero, as desired. □

2/22/2022

Lecture 11

Cauchy's Theorem Applications I

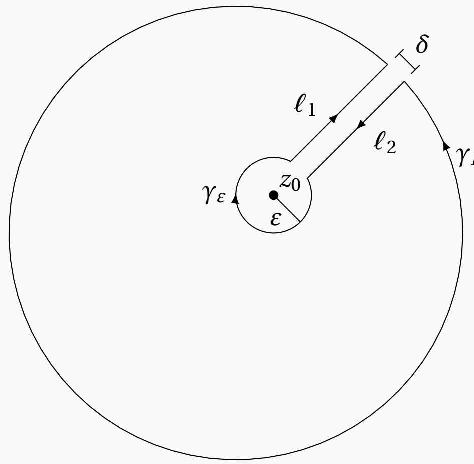
The following example was most likely not covered in lecture (as I missed it), but it helps in deriving Cauchy's integral formula; I believe the integral over $\oint_{\partial D_3(i)} \frac{1}{z}\,\mathrm{d}z$ was covered instead. I don't particularly like the proof given in the professor's notes, though, so I'm providing a different one here, roughly taken from the textbook's proof of Cauchy's integral formula.

Example 11.1: Contour integral of $\frac{1}{z}$

Suppose D is a disk, where ∂D denotes its boundary. Further, suppose $z_0 \in D$; we'd like to compute

$$\oint_{\partial D} \frac{1}{z - z_0}\,\mathrm{d}z.$$

To do so, let us look at the keyhole contour:



Here, we let γ_R denote the outer arc of radius R, γ_ε denote the inner arc of radius ε, and ℓ_1, ℓ_2 denote the two line segments connecting the arcs, separated by a distance δ.

Firstly, notice that by Cauchy's theorem

$$\oint_\gamma \frac{1}{z-z_0}\,\mathrm{d}z = 0,$$

since $\frac{1}{z-z_0}$ is holomorphic along the curve and its interior (here, γ denotes the entire keyhole). We can split this up into several parts as well:

$$0 = \int_{\gamma_R} \frac{1}{z-z_0}\,\mathrm{d}z + \int_{\gamma_\varepsilon} \frac{1}{z-z_0}\,\mathrm{d}z + \int_{\ell_1} \frac{1}{z-z_0}\,\mathrm{d}z + \int_{\ell_2} \frac{1}{z-z_0}\,\mathrm{d}z.$$

Notice that as $\delta \to 0$, because $\frac{1}{z-z_0}$ is continuous, we have

$$\int_{\ell_1} \frac{1}{z-z_0}\,\mathrm{d}z + \int_{\ell_2} \frac{1}{z-z_0}\,\mathrm{d}z = 0,$$

since the two lines are of opposite orientation.

This means that we're left with

$$0 = \oint_{\partial D} \frac{1}{z-z_0}\,\mathrm{d}z + \oint_{\partial D_\varepsilon} \frac{1}{z-z_0}\,\mathrm{d}z,$$

where the arcs γ_R and γ_ε converge onto circles $\partial D_R = \partial D$ (from the original integral) and ∂D_ε.

The first integral is what we want, and we can rewrite the second integral over ∂D_ε by shifting it with $w = z - z_0$; this makes D_ε now centered at the origin, and we can use the typical parameterization (negated since the typical parameterization goes counterclockwise):

$$\begin{aligned}\oint_{\partial D_\varepsilon} \frac{1}{z-z_0}\,\mathrm{d}z = \oint_{\partial D_\varepsilon(0)} \frac{1}{w}\,\mathrm{d}w &= -\int_0^{2\pi} \frac{1}{\varepsilon e^{-it}} \cdot \varepsilon i e^{-it}\,\mathrm{d}t \\ &= -\int_0^{2\pi} i\,\mathrm{d}t \\ &= -2\pi i\end{aligned}$$

Plugging this in, we have

$$0 = \oint_{\partial D} \frac{1}{z-z_0}\,\mathrm{d}z - 2\pi i \implies \oint_{\partial D} \frac{1}{z-z_0}\,\mathrm{d}z = 2\pi i.$$

2/24/2022

Lecture 12

Cauchy Theorem Applications II

Last time we considered $e^{-\pi x^2}$ as a probability density function. That is, if we integrate over the real axis, we get 1. We also considered $\cos(2\pi ax)$ for $x \in \mathbb{R}$. We claimed that the expectation of this function

$$\mathbb{E}[\cos(2\pi ax)] = \int_{\mathbb{R}} \cos\big(2piax\big) e^{-\pi x^2}\, \mathrm{d}x = e^{-\pi a^2},\ a \in \mathbb{R}.$$

To do this, we looked at

$$e^{2\pi axi} = \cos(a\pi ax) + i\sin(a\pi ax).$$

Since sin is an odd function, and $e^{-\pi x^2}$ is an even function, their product becomes an odd function, and the integral evaluates to zero on the real axis.

Further, looking at

$$\int_{\mathbb{R}} e^{2\pi axi} e^{-pix^2}\, \mathrm{d}x,$$

we consider the holomorphic function $f(z) = e^{-\pi z^2}$, and its contour integral over the rectangle above the x axis with sides at $-R$ and R, intersecting the imaginary axis at ai.

Since f is holomorphic on the entire plane, its contour integral along this closed loop must be zero.

Along the real axis from $-R$ to R, we have

$$\begin{aligned}\int_{\gamma_0} f(z)\, \mathrm{d}z &= \int_{-R}^{R} e^{-\pi x^2}\, \mathrm{d}x \\ &\to 1 && (R \to \infty)\end{aligned}$$

Along the horizontal line ai, we can parameterize it as

$$z(t) = (-t) + ai,\ t \in [-R, R].$$

This means that we have

$$\int_{\gamma_a} f(z)\, \mathrm{d}z = -\int_{-R}^{R} e^{-\pi((-t)+ai)^2}\, \mathrm{d}t$$

The exponent can be simplified as

$$-\pi((-t)+ai)^2 = -\pi(t^2 - 2ati - a^2) = -\pi t^2 + 2\pi ati + \pi a^2.$$

This integral then becomes

$$\begin{aligned}\int_{\gamma_a} f(z)\, \mathrm{d}z &= -\int_{-R}^{R} e^{-\pi t^2} e^{2\pi ati} e^{\pi a^2}\, \mathrm{d}t \\ &= -e^{\pi a^2} \int_{-R}^{R} e^{-\pi t^2} e^{2\pi ati}\, \mathrm{d}t\end{aligned}$$

It turns out that the integral is exactly what we want to know (as $R \to \infty$), and we have a constant multiple at the beginning.

Looking at γ_1 from R to $R + ai$, we claim that the integrals over γ_1 (and similarly γ_2 from $-R + ai$ to $-R$) evaluate to zero as $R \to \infty$.

We can parameterize γ_1 as $z(t) = R + it$ for $t \in [0, a]$. The integral then becomes

$$\int_{\gamma_1} f(z)\, \mathrm{d}z = i\int_0^a e^{-\pi(R+it)^2}\, \mathrm{d}t$$

Again expanding the exponent, we have

$$-\pi(R+it)^2 = -\pi(R^2+2Rti-t^2) = -\pi R^2 - 2\pi Rti + \pi t^2.$$

This means the integral becomes

$$\begin{aligned}\int_{\gamma_1} f(z)\,\mathrm{d}z &= i\int_0^a e^{-\pi R^2} e^{-2\pi Rti} e^{\pi t^2}\,\mathrm{d}t \\ &= ie^{-\pi R^2}\int_0^a e^{-2\pi Rti} e^{\pi t^2}\,\mathrm{d}t\end{aligned}$$

We have $e^{-\pi R^2}$ in the front, which goes to zero as $R \to \infty$. This means that if the integral has a bounded norm, then it will always go to zero as $R \to \infty$. We have

$$\left|\int_0^a e^{-\pi 2Rti} e^{\pi t^2}\,\mathrm{d}t\right| \le \left(\sup_{t\in[0,a]}\left|e^{-\pi 2Rti} e^{\pi t^2}\right|\right)\cdot a$$

Since $e^{-\pi 2Rti}$ is always on the unit circle, its norm is one. Further, $e^{\pi t^2}$ is a real function, and is also an increasing function. This means that we achieve our maximum at $t = a$. Together, we have

$$= e^{\pi a^2}\cdot a$$

As such, the integral is bounded, since a is fixed. This means the original integral goes to zero as $R \to \infty$.

γ_2 is similar, and also converges to zero as $R \to \infty$.

Putting everything together now, we have

$$\begin{aligned}0 &= 1 - e^{\pi a^2}\int_{-R}^R e^{2\pi ati} e^{-\pi t^2}\,\mathrm{d}t + 0 + 0 \\ e^{\pi a^2}\int_{-R}^R e^{2\pi ati} e^{-\pi t^2}\,\mathrm{d}t &= 1 \\ \int_{-R}^R e^{2\pi ati} e^{-\pi t^2}\,\mathrm{d}t &= e^{-\pi a^2}\end{aligned}$$

As another example, suppose we consider

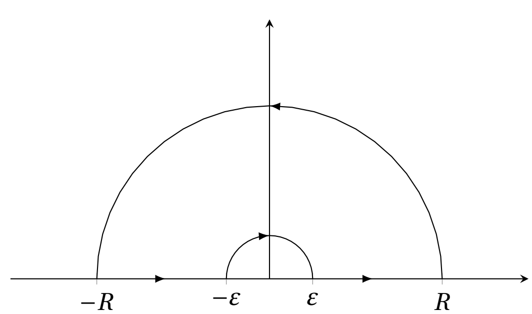
$$\int_0^\infty \frac{1-\cos t}{t^2}\,\mathrm{d}t = \frac{\pi}{2}.$$

This function has no simple antiderivative, so we will try consider a complex function to help us integrate.

By Euler's formula, we have $e^{zi} = \cos(z) + i\sin(z)$. Since $\sin(z)$ contributes no extra value (since it is odd), we can consider

$$f(z) = \frac{1-e^{iz}}{z^2}.$$

Further, we consider the indented semicircle:



Later, we want $R \to +\infty$ and $\varepsilon \to 0^+$. The part along the real axis is exactly what we want, and the sin function cancels with itself (as it is an odd function).

Let us denote the inner semicircle as γ_ε and the outer semicircle as γ_R. Further, let us call the part of the curve along the positive real axis as γ_+ and the part along the negative real axis as γ_-.

By Cauchy's theorem, the sum of the integrals along these four pieces is zero. As before, let us calculate the integrals one by one.

Let's start with γ_+ and γ_-, since these should give us the real integral that we want to know. We can parameterize these curves as

$$\begin{aligned} z_-(t) &= t,\ t \in [-R, -\varepsilon] \\ z_+(t) &= t,\ t \in [\varepsilon, R] \end{aligned}$$

This means that we have

$$\begin{aligned} \int_{\gamma_-} f(z)\,\mathrm{d}z + \int_{\gamma_+} f(z)\,\mathrm{d}z &= \int_{-R}^{-\varepsilon} \frac{1-e^{it}}{t}\,\mathrm{d}t + \int_{\varepsilon}^{R} \frac{1-e^{it}}{t}\,\mathrm{d}t j \\ &= \int_{-R}^{-\varepsilon} \frac{1-\cos t}{t^2}\,\mathrm{d}t + \int_{\varepsilon}^{R} \frac{1-\cos t}{t^2}\,\mathrm{d}t - i\left(\int_{-R}^{-\varepsilon} \frac{\sin t}{t^2}\,\mathrm{d}t + \int_{\varepsilon}^{R} \frac{\sin t}{t^2}\,\mathrm{d}t\right) \end{aligned}$$

Since $\sin t$ is an odd function and t^2 is an even function, the imaginary part will go to zero, since the two integrals together are symmetric about 0. This means that the contribution from one integral cancels out with the contribution from the other. As such, the imaginary part is zero. Further, since $\cos t$ is even and t^2 is also even, the contribution from both integrals are the same:

$$= 2\int_{\varepsilon}^{R} \frac{1-\cos t}{t^2}\,\mathrm{d}t$$

As $\varepsilon \to 0$ and $R \to \infty$, we have

$$\int_{\gamma_-} f(z)\,\mathrm{d}z + \int_{\gamma_+} f(z)\,\mathrm{d}z = 2\int_0^{\infty} \frac{1-\cos t}{t^2}\,\mathrm{d}t.$$

Now let us look at γ_R.

We can parameterize the curve as $z(t) = Re^{it}$ for $t \in [0, \pi]$. The integral becomes

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = \int_0^{\pi} \frac{1-e^{i(Re^{it})}}{(Re^{it})^2} \cdot Rie^{it}\,\mathrm{d}t.$$

This integral is quite complicated, so let's try another approach; let us bound the norm of the integral:

$$\begin{aligned} \left|\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z\right| &\le \sup_{z\in\gamma_R} \left|\frac{1-e^{iz}}{z^2}\right| \cdot \operatorname{length}(\gamma_R) \\ &= \sup_{z\in\gamma_R} \left|\frac{1-e^{iz}}{z^2}\right| \cdot \pi R \end{aligned}$$

We can rewrite this norm as

$$\left|\frac{1-e^{iz}}{z^2}\right| = \left|\frac{1}{z^2} - \frac{e^{iz}}{z^2}\right| \le \frac{1}{|z|^2} + \frac{\left|e^{iz}\right|}{|z|^2}.$$

Since z is on γ_R, we know that $|z| = R$.

Looking at $\left|e^{iz}\right|$, suppose we have $z = x + iy$ for $x, y \in \mathbb{R}$. This means that we have

$$\left|e^{iz}\right| = \left|e^{ix-y}\right| = \left|e^{ix}e^{-y}\right| = e^{-y}.$$

Since $y \geq 0$ (as z is on the semicircle above the real axis), we know that $e^{-y} \leq 1$. Plugging these values in, we have

$$\left|\frac{1-e^{iz}}{z^2}\right| \leq \frac{1}{R^2} + \frac{1}{R^2} = \frac{2}{R^2}.$$

In the original bound again, we have

$$\sup_{z \in \gamma_R}\left|\frac{1-e^{iz}}{z^2}\right| \cdot \pi R \leq \frac{2}{R^2} \cdot \pi R = \frac{2\pi}{R} \to 0,$$

as $R \to \infty$.

Lastly, we can look at the smaller semicircle. It turns out that this will actually give us some contribution, because of the singularity at 0.

From before, parameterizing immediately will not be easy to deal with. As such, suppose we look at the power series expansion of e^{iz}. This means we have

$$\begin{aligned}
e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots \\
1 - e^{iz} &= -iz - \frac{(iz)^2}{2!} - \frac{(iz)^3}{3!} - \cdots \\
\frac{1-e^{iz}}{z^2} &= \frac{-iz - \frac{(iz)^2}{2!} - \frac{(iz)^3}{3!} - \cdots}{z^2} \\
&= -\frac{i}{z} + \frac{1}{2!} + O(z) + O(z^2) + \cdots
\end{aligned}$$

(Here, $O(f)$ represents some constant multiple of the function f)

Since the original power series for e^{iz} is convergent, then this final power series is also convergent. As $z \to 0$, every term after $\frac{1}{2} + O(z) + \cdots$ converges to zero as well (let us call this tail $g(z)$). This means that we're just left with $-\frac{i}{z}$.

As such, we have

$$\frac{1-e^{iz}}{z^2} = -\frac{i}{z} + g(z),$$

where g is holomorphic with $\lim_{z\to 0} g(z) = 0$.

The integral then becomes

$$\int_{\gamma_\varepsilon} -\frac{i}{z} + \int_{\gamma_2} \mathrm{d}z.$$

We can see that the second integral goes to zero, since the integral is bounded, and its length goes to zero as $\varepsilon \to 0$. (As such the supremum of the integrand multiplied by the length of the arc goes to zero.)

$$\left|\int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z\right| \leq \underbrace{\sup_{z\in\gamma_\varepsilon}\left|g(z)\right|}_{\text{bounded}} \cdot \underbrace{\text{length}(\gamma_\varepsilon)}_{\to 0}.$$

This means we're just left with the integral over $-\frac{i}{z}$. Now parameterizing γ_ε as $z(t) = \varepsilon e^{-i(\pi - t)}$ for $t \in [0, \pi]$, we have

$$\begin{aligned}
\int_{\gamma_\varepsilon} -\frac{i}{z}\,\mathrm{d}z &= \int_0^\pi -\frac{i}{\varepsilon e^{i(pi-t)}} \cdot \varepsilon e^{i(pi-t)}(-i)\,\mathrm{d}t \\
&= \int_0^\pi (-i)(-i)\,\mathrm{d}t \\
&= -\int_0^\pi \mathrm{d}t = -\pi
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
2\int_0^\infty \frac{1-\cos t}{t^2}\,\mathrm{d}t + 0 - \pi &= 0 \\
\int_0^\infty \frac{1-\cos t}{t^2}\,\mathrm{d}t &= \frac{\pi}{2}
\end{aligned}$$

12.1 Cauchy's Integral Formula

As motivation, let us consider the following example.

Example 12.1

Suppose we have a power series with z_0 as its center. Before now, we've always taken 0 as the center, but we can just shift it to make it centered at z_0. That is, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Suppose we have a disk D fully inside the radius of convergence of the power series, centered at z_0.

Let us look at the integral

$$\oint_{\partial D} \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

Note that f is holomorphic in D, and $z - z_0 \neq 0$ along the boundary of the disk. This means that the entire integrand $\frac{f(z)}{z-z_0}$ is holomorphic along the boundary (but not within the boundary, specifically at z_0). This means it makes sense to talk about this integral.

If we expand out $f(z)$, we have

$$\begin{aligned} f(z) &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \\ \frac{f(z)}{z - z_0} &= \frac{a_0}{z - z_0} + \underbrace{a_1 + a_2(z - z_0) + a3(z - z_0)^3}_{g(z)} \end{aligned}$$

The tail here ($g(z)$) is holomorphic, and as such its integral evaluates to zero along the boundary. This means that we have

$$\begin{aligned} \oint_{\partial D} \frac{f(z)}{z - z_0} \, \mathrm{d}z &= \oint_{\partial D} \frac{a_0}{z - z_0} \, \mathrm{d}z + \oint_{\partial D} g(z) \, \mathrm{d}z \\ &= a_0 \oint_{\partial D} \frac{1}{z - z_0} \, \mathrm{d}z \\ &= 2\pi i a_0 = 2\pi i f(z_0) \end{aligned}$$

This means that we can rearrange to get

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

This is actually a special case of the Cauchy's integral formula—it turns out that this holds not only for power series; as long as f is holomorphic in the disk D, then this formula holds. Further, it also holds for larger powers in the denominator (i.e. $(z - z_0)^n$).

Theorem 12.2: Cauchy's Integral Formula ($n = 0$)

Suppose Ω is an open set in $\mathbb{C}$ and f is holomorphic on Ω. Then, for any disk D such that $\overline{D} \subseteq \Omega$, we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z - z_0} \, \mathrm{d}z,$$

for any $z_0 \in D$.

Proof. Looking at the RHS, we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)}{z-z_0}\,\mathrm{d}z &= \frac{1}{2\pi i}\oint_{\partial D}\frac{f(z_0)+\big(f(z)-f(z_0)\big)}{z-z_0}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\oint_{\partial D}\frac{f(z_0)}{z-z_0}\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z \\ &= f(z_0) + \frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z\end{aligned}$$

Here, the last equality can be derived from Example 11.1, and we just need to show that the second term vanishes.

Suppose we take $\varepsilon > 0$ such that $\overline{D_\varepsilon(z_0)} \subset D$. We then have

$$f'(z_0) = \frac{f(z)-f(z_0)}{z-z_0} + \phi_\varepsilon(z),$$

for $\lim_{\varepsilon\to 0^+}\sup_{z\in\partial D_\varepsilon}\left|\phi_\varepsilon(z)\right| = 0$.

We can rearrange to find

$$f(z)-f(z_0) = f'(z_0)(z-z_0) + \phi_\varepsilon(z)(z-z_0).$$

Firstly, since $\frac{f(z)-f(z_0)}{z-z_0}$ is holomorphic on $\Omega\setminus\{z_0\}$, the integral over ∂D will be equal to the integral over $\partial D_\varepsilon(z_0)$. We can then plug the earlier result in to find that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z &= \frac{1}{2\pi i}\oint_{\partial D_\varepsilon(z_0)}\frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\oint_{\partial D_\varepsilon(z_0)} f'(z_0)+\phi_\varepsilon(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\oint_{\partial D_\varepsilon(z_0)} \phi_\varepsilon(z)\,\mathrm{d}z\end{aligned}$$

Here, the last equality is due to the fact that $f'(z)$ is holomorphic on Ω; by Cauchy's theorem, the integral vanishes, leaving just the integral of $\phi_\varepsilon(z)$.

We can now look at the norm to see that

$$\left|\frac{1}{2\pi i}\oint_{\partial D_\varepsilon(z_0)}\phi_\varepsilon(z)\,\mathrm{d}z\right| \le \varepsilon\cdot\sup_{z\in\partial D_\varepsilon(z_0)}\left|\phi_\varepsilon(z)\right|.$$

As $\varepsilon \to 0$, the RHS tends to zero; this means that

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z = 0.$$

□

3/1/2022

Lecture 13

Cauchy's Integral Formula

Example 13.1

Calculate $\oint_{\partial D_3(0)} \frac{e^z+z^2+3}{z-2}\,\mathrm{d}z$.

Here, we can recognize that the denominator falls into the form of Cauchy's integral formula, and that $2 \in D_3(0)$. This means that we have

$$\frac{1}{2\pi i}\oint_{\partial D_3(0)} \frac{e^z+z^2+3}{z-2}\,\mathrm{d}z = \left(e^z+z^2+3\right)\Big|_{z=2} = e^2 = 7.$$

This means that

$$\oint_{\partial D_3(0)} \frac{e^z+z^2+3}{z-2}\,\mathrm{d}z = 2\pi i(e^2+7).$$

Example 13.2

Calculate $\oint_{\partial D_1(0)} \frac{e^z+z^2+3}{z-2}\,\mathrm{d}z$.

Here, we notice that $2 \notin D_1(0)$, but this also means that $\frac{e^z+z^2+3}{z-2}$ is holomorphic on $D_1(0)$, and also holomorphic on $D_{1.5}(0) \supset \overline{D_1(0)}$. Utilizing Cauchy's theorem, we have

$$\oint_{\partial D_1(0)} \frac{e^z+z^2+3}{z-2}\,\mathrm{d}z = 0.$$

Theorem 13.3: Cauchy's Integral Formula

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, and f is holomorphic over Ω. Since f is holomorphic, it has infinitely many complex derivatives over Ω. Moreover, for any disk D with $\overline{D} \subset \Omega$ and any $z_0 \in D$, the nth derivative at z_0 is calculated as

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\partial D} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

Proof. We can prove this formula by inducting. As a base case of $n = 0$, we've already shown that for any $z_0 \in \Omega$,

$$f^{(0)}(z_0) = \frac{0!}{2\pi i}\oint_{\partial D} \frac{f(z)}{(z-z_0)^{0+1}}\,\mathrm{d}z.$$

Suppose that the formula holds for $n = k$; we need to show that the formula holds for $f^{(k+1)}(z_0)$.

Since Ω is open, for any $z_0 \in \Omega$ there exists some disk D that contains z_0 such that $\overline{D} \subset \Omega$.

By definition, we have

$$f^{(k+1)}(z_0) = \lim_{h\to 0} \frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h}.$$

For $|h|$ small enough such that $z_0 + h \in D$, we can apply the IH to say

$$\begin{aligned}
&\frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h} \\
&= \frac{k!}{2\pi i h} \oint_{\partial D} f(z)\left(\frac{1}{(z-(z+h))^{k+1}} - \frac{1}{(z-z_0)^{k+1}}\right) \mathrm{d}z \\
&= \frac{k!}{2\pi i h} \oint_{\partial D} f(z)\left(\frac{(z-z_0)^{k+1} - (z-(z_0+h))^{k+1}}{(z-(z_0+h))^{k+1}(z-z_0)^{k+1}}\right) \mathrm{d}z
\end{aligned}$$

Utilizing the formula $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$, we have

$$\begin{aligned}
&= \frac{k!}{2\pi i h} \oint_{\partial D} f(z)\left(\frac{(z-z_0-z+z_0+h)\big((z-z_0)^k + (z-z_0)^{k-1}(z-(z_0+h)) + \cdots + (z-(z_0+h))^k\big)}{(z-(z_0+h))^{k+1}(z-z_0)^{k+1}}\right) \mathrm{d}z \\
&= \frac{k!}{2\pi i} \oint_{\partial D} f(z)\left(\frac{(z-z_0)^k + (z-z_0)^{k-1}(z-(z_0+h)) + \cdots + (z-(z_0+h))^k}{(z-(z_0+h))^{k+1}(z-z_0)^{k+1}}\right) \mathrm{d}z
\end{aligned}$$

Taking the limit as $h \to 0$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h} &= \frac{k!}{2\pi i} \oint_{\partial D} f(z)\left(\frac{(z-z_0)^k + (z-z_0)^{k-1}(z-z_0) + \cdots + (z-z_0)^k}{(z-z_0)^{k+1}(z-z_0)^{k+1}}\right) \mathrm{d}z \\
&= \frac{k!}{2\pi i} \oint_{\partial D} f(z)\left(\frac{(k+1)(z-z_0)^k}{(z-z_0)^{2k+2}}\right) \mathrm{d}z \\
&= \frac{k!}{2\pi i} \oint_{\partial D} f(z)\left(\frac{k+1}{(z-z_0)^{k+2}}\right) \mathrm{d}z \\
&= \frac{(k+1)!}{2\pi i} \oint_{\partial D} \frac{f(z)}{(z-z_0)^{k+2}} \mathrm{d}z
\end{aligned}$$

As such, by the principles of induction, we've just shown that for any $n = 0, 1, \ldots$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\partial D} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z.$$

□

Example 13.4

Use Cauchy's integral formula to compute

$$\oint_{\partial D_3(0)} \frac{e^z + z^2 + 3}{(z-2)^2} \mathrm{d}z.$$

Since $2 \in D_3(0)$, we have

$$\frac{1}{2\pi i} \oint_{\partial D_3(0)} \frac{e^z + z^2 + 3}{(z-2)^2} \mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}z}\left(e^z + z^2 + 3\right)\Big|_{z=2} = \left(e^z + 2z\right)\big|_{z=2} = e^2 + 4.$$

As such,

$$\oint_{\partial D_3(0)} \frac{e^z + z^2 + 3}{(z-2)^2} \mathrm{d}z = 2\pi i\left(e^2 + 4\right).$$

A corollary of Cauchy's integral formula is Cauchy's inequality.

Corollary 13.5: Cauchy's Inequality

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, and f is holomorphic over Ω. Then, for each disk centered at $z_0 \in \Omega$ with $\overline{D_R(z_0)} \subset \Omega$, we have

$$\left|f^{(n)}(z_0)\right| \le \frac{n!\|f\|_{\partial D_R(z_0)}}{R^n}.$$

Here, $\|f\|_{\partial D_R(z_0)} \coloneqq \sup_{z\in \partial D_R(z_0)} \left|f(z)\right|$.

Proof. By Cauchy's integral formula, we have

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\oint_{\partial D_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z\right| \\
&\le \frac{n!}{2\pi i}\sup_{z\in\partial D_R(z_0)}\left|\frac{f(z)}{(z-z_0)^{n+1}}\right|\cdot \operatorname{length}(\partial D_R(z_0)) \\
&= \frac{n!}{2\pi i}\sup_{z\in\partial D_R(z_0)}\frac{\left|f(z)\right|}{R^{n+1}}\cdot 2\pi R i \\
&= \frac{n!}{R^n}\sup_{z\in\partial D_R(z_0)}\left|f(z)\right| \\
&= \frac{n!}{R^n}\|f\|_{\partial D_R(z_0)}
\end{aligned}$$

□

A corollary of Cauchy's inequality is Liouville's theorem.

Theorem 13.6: Louiville's Theorem

A bounded entire function can only be constant.

Proof. Suppose f is an entire function with $\left|f(z)\right| \le M$ for any $z \in \mathbb{C}$, and $M > 0$ is a constant.

We then have from Cauchy's inequality that for any $z_0 \in \mathbb{C}$,

$$\left|f'(z_0)\right| \le \frac{\|f\|_{\partial D_R(z_0)}}{R},$$

for any $R \in \mathbb{R}$. As such, if we take the limit as $R \to +\infty$, the numerator is always bounded by M, while the denominator increases to ∞, making the entire RHS go to zero.

This means that $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$. By Lemma 9.11, we know that $f(z)$ must be a constant function, as desired. □

3/3/2022

Lecture 14

Fundamental Theorem of Algebra, Analytic Functions

Let's finish the proof of the fundamental theorem of algebra.

Theorem 14.1: Fundamental Theorem of Algebra

If we have a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ for $a_i \in \mathbb{C}$, and in particular, $a_n \neq 0$. n is then $\deg(p)$.

Further, suppose $n = \deg(p) \geq 1$, i.e. p is not constant.

Then, $p(z) = 0$ has n roots, counting multiplicities.

Proof. Last time we sketched the proof.

First, we prove $p(z) = 0$ has at least one root in $\mathbb{C}$. Suppose for contradiction this is not the case. This means that for any $z \in \mathbb{C}$, $p(z) \neq 0$; let us then consider the reciprocal $f(z) = \frac{1}{p(z)}$. Since p has no roots, f is defined everywhere, and as such is holomorphic on $\mathbb{C}$.

We can also observe that this function $f(z)$ is bounded for some $C > 0$. That is, we can write

$$p(z) = a_n z^n \left(1 + \underbrace{\frac{a_{n-1}}{a_n} \frac{1}{z} + \frac{a_{n-2}}{a_n} \frac{1}{z^2} + \cdots}_{g(z)} \right).$$

As $|z| > R$ grows to infinity, $\left|g(z)\right| \to 0$, since we have $\frac{1}{z^i}$ in every term. Formally, there exists some $R > 0$ such that $|z| > R$ implies that $\left|g(z)\right| < \frac{1}{2}$. This means $\left|1 + g(z)\right| \geq 1 - \left|g(z)\right| \geq 1 - \frac{1}{2} = \frac{1}{2}$.

Putting this together, we have

$$\left|f(z)\right| = \frac{1}{\left|p(z)\right|} = \frac{1}{|a_n||z|^n} \cdot \frac{1}{\left|1 + g(z)\right|} \leq \frac{2}{|a_n| R^n},$$

which is a finite constant. This means that we have found a bound for $\left|f(z)\right|$ for $z > R$. Now, let us consider the closed disk $\overline{D_R(0)}$. Since $f(z)$ is holomorphic, it is continuous on $\overline{D_R(0)}$. Since this disk is closed and bounded, it is compact. We also know that any continuous function on a compact set must also be bounded—this means that $\left|f(z)\right|$ is also bounded on $\overline{D_R(0)}$. With the previous result, we now know that $f(z)$ is bounded on all of $\mathbb{C}$.

As such, by Liouville's theorem, we know that $f(z)$ is a constant function. Since $f(z) = \frac{1}{p(z)}$, this now implies that $p(z)$ is also constant, which is a contradiction.

Now, we will show that $p(z) = 0$ has n roots. From the first result, we know that there is some $z_1 \in \mathbb{C}$ such that $p(z_1) = 0$.

We can then use the division algorithm to write $p(z) = q(z)(z - z_1) + r(z)$. This remainder must be a constant, since $\deg(r) < \deg(z - z_1) = 1$. Further, we must have $p(z_1) = q(z_1)(z - z_1) + r(z_1) = 0$, forcing $r(z_1) = 0$.

This means that after finding a root z_1, we can factor p into $q(z)$ and $(z - z_1)$, where $\deg(q) = \deg(p) - 1$. We can keep going, and look at $q(z)$.

If $q(z)$ has degree 0, then it is a constant, and our only root is z_1. Otherwise, we can apply the same logic to find $n - 1$ roots of $q(z)$. After a finite number of steps, we now can write

$$p(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n).$$

□

The scope of the midterm ends here.

14.1 Analytic Functions

Recall that a power series is defined as $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, with a convergence radius R. This means that the power series converges on the disk $D_R(z_0)$. We've also shown in a homework problem that this power series can be shifted to be centered at any other point in the disk, and it will still be convergent on some smaller radius R'.

Definition 14.2: Analytic Function

For an open set $\Omega \subseteq \mathbb{C}$, and a function f over Ω, f is *analytic over* Ω if for any point $z_0 \in \Omega$, there is a disk centered at z_0 such that $f(z)$ can be written as a convergent power series over this disk.

Corollary 14.3

Any power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic over $D_R(z_0)$, where R is its convergence radius.

Theorem 14.4

For an open set $\Omega \subseteq \mathbb{C}$, any holomorphic function f over Ω is analytic.

Proof. We need to show that for any $z_0 \in \Omega$, there is some $R > 0$ with $D_R(z_0) \subseteq \Omega$ (and $\overline{D_R(z_0)} \subseteq \Omega$) such that $f(z)$ can be written as a power series centered at z_0 on $D_R(z_0)$.

For any $z \in D_R(z_0)$, we have by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(w)}{w-z} \, \mathrm{d}w.$$

Note that we can write

$$w - z = (w-z_0) - (z-z_0) = (w-z_0)\left(1 - \frac{z-z_0}{w-z_0}\right).$$

Since $w \in \partial D_R(z_0)$, the denominator is never zero (as it is always the case that $w \neq z_0$). Further, we can show that $\left|\frac{z-z_0}{w-z_0}\right| < 1$. Since $z \in D_R(z_0)$ and $w \in \partial D_R(z_0)$, it must be the case that $|z-z_0| < |w-z_0|$, meaning $\left|\frac{z-z_0}{w-z_0}\right| < 1$.

We can now rewrite the second term as a geometric series when placed in the denominator of a fraction:

$$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n.$$

Plugging this into the integral formula, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(w)}{w-z} \, \mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D_R(z_0)} f(w) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} \, \mathrm{d}w \end{aligned}$$

We can swap the integral and summation here because the integral is over a compact domain and the power series is convergent:

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left(\int_{\partial D_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} \, \mathrm{d}w \right) (z-z_0)^n$$

We're done here; this is a power series. However, we can keep going a little bit here and simplify to find what the coefficients are, by Cauchy's Integral formula again (the $n!$ is missing, so we have to divide it out here):

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

□

As a remark, we've shown that holomorphicity and (complex) analyticity are equivalent. However, over the reals, analyticity only implies differentiability, but not the other way around.

Theorem 14.5: Unique Continuation Theorem

Suppose we have an open set $\Omega \subseteq \mathbb{C}$, and a function f holomorphic on Ω. If there is a convergent sequence $\{z_n\}$ with distinct points in Ω (i.e. $\{z_n\}$ is not the trivial constant sequence) such that $f(z_n) = 0$, then $f(z) = 0$ everywhere on Ω.

Proof. Suppose we have a convergent series $\{z_n\}$ in Ω, such that $z_n \to z_0 \in \Omega$. Further, suppose $f(z_n) = 0$, for the holomorphic function f on $Omega$. Since f is holomorphic, it is also continuous. This means that

$$f(z_0) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = 0.$$

We've shown that all holomorphic functions are analytic, so let us write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on a small disk with z_0 as its center.

Suppose for contradiction that f is not zero on $D(z_0)$. This means that there is some nonzero coefficient a_k of the power series, where $k > 0$; in particular, let k be the smallest index of such a coefficient. This means that

$$\begin{aligned} f(z) &= a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \cdots & (a_k \neq 0) \\ &= a_k(z-z_0)^k + \left(1 + \underbrace{\frac{a_{k+1}}{a_k}(z-z_0) + \frac{a_{k+2}}{a_k}(z-z_0)^2 + \cdots}_{g(z)}\right) \end{aligned}$$

As $z \to z_0$, then $1 + g(z) \to 1$, and for $\{z_n\}$, when n is big enough, $1 + g(z_n) \neq 0$.

Evaluating at z_n, then $f(z_n) = a_k(z_n - z_0)^k(1 + g(z_n))$. We know that $f(z_n) = 0$, and $a_k \neq 0$, $z_n \neq z_0$ (as $\{z_n\}$ are all distinct points), and $1 + g(z_n) \neq 0$. This is thus a contradiction; the LHS is zero but the RHS cannot be zero.

The only way to resolve this contradiction is to take f as constant zero, as desired. □

Definition 14.6: Extension

Suppose $U \subset V$ are two sets. For the two functions $f : U \to \mathbb{C}$ and $\tilde{f} : V \to \mathbb{C}$ with $\tilde{f}\big|_U = f$, i.e. $\tilde{f}(z) = f(z)$ for all $z \in U$, then we say that $\tilde{f}$ is an *extension* of f from U to V.

Moreover, when U and V are both open subsets of $\mathbb{C}$, and $f, \tilde{f}$ are both analytic (or holomorphic), we say that $\tilde{f}$ is an *analytic extension* (or *holomorphic extension*), or an *analytic continuation* (or *holomorphic continuation*) of f from U to V.

Corollary 14.7

Suppose we have open sets $U \subseteq V \subseteq \mathbb{C}$, and let f be a holomorphic function on U.

Then, f can have at most one holomorphic extension to V (it is possible that f cannot be extended at all).

Proof. Suppose we have two different extensions g_1, g_2 on V. This means that $g_1(z) - g_2(z) = 0$ on U, since both are extensions of f on U.

However, by the unique continuation theorem, $g_1(z) - g_2(z)$ must also be zero on V (as we can find some convergent series $\{z_n\}$ on $U \subseteq V$ on which $g_1(z) - g_2(z) = 0$). This is a contradiction, since we've assumed that g_1 and g_2 are distinct. □

One other note is that if we use "smooth" rather than "holomorphic (analytic)", then there always exist infinitely many different extensions—this is another big difference between smooth functions and holomorphic functions.

Example 14.8

Consider the function

$$g(x) = \begin{cases} e^{\frac{1}{1-x^2-y^2}} & x^2+y^2 \geq 1 \\ 0 & x^2+y^2 < 1 \end{cases}.$$

This is a smooth extension of the constant zero function from the unit disk in $\mathbb{R}^2$.

3/10/2022

Lecture 15

Morera's Theorem, Sequences of Holomorphic Functions

We'll talk some more about different applications of Cauchy's theorem. First is Morera's Theorem, which is the reverse of Goursat's theorem (Theorem 10.3); it provides a characterization for holomorphicity.

Theorem 15.1: Morera's Theorem

Suppose f is a continuous function over an open disk D in $\mathbb{C}$. If for each triangle T with $\overline{T} \subset D$, the contour integral of f over the boundary vanishes, i.e.

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0,$$

then f must be holomorphic over the disk D.

Proof. We show that f is holomorphic by showing that it's the derivative of some holomorphic function F.

Recall from the proof of Cauchy's theorem (Theorem 10.1) that we use Goursat's theorem (Theorem 10.3), which says

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

for any triangle T, to construct a primitive for the function f. Here, we may use the same construction to obtain $F\colon D \to \mathbb{C}$.

Then, we know that F is holomorphic with derivative $F' = f$. Since F is also analytic, we know that F has any order complex derivative; specifically, f is holomorphic. □

An application of Morera's theorem is with convergent sequences of holomorphic functions.

Firstly, here are some remarks on a famous construction by Weierstrass for real-valued functions:

- In real analysis, the Weierstrass Approximation Theorem states

Theorem 15.2: Weierstrass Approximation Theorem

For any continuous function f over $[a,b] \subset \mathbb{R}$, given any $\varepsilon > 0$, there exists a polynomial p such that $\|f - p\|_{C^0([a,b])} < \varepsilon$.

Equivalently, for any continuous real function f on $[a,b]$, there is a sequence of polynomial functions $\{p_n\}$ that uniformly converges to f.

- On the other hand, Weierstrass constructed examples of such functions, now referred to as Weierstrass functions:
$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right),$$
for $0 < b < 1$ a positive odd integer, and $ab > 1 + \frac{3\pi}{2}$.

  These are real-valued functions that are continuous everywhere but differentiable nowhere.

- These two discoveries together indicate that for real-valued functions, convergence (even uniform convergence over a compact set) may cause the loss of differentiability.

However, the situation is different for complex-valued functions—holomorphicity will be preserved under a limiting process. Precisely, we have the following theorem:

Theorem 15.3: Preservation of holomorphicity with a limit

Suppose Ω is an open subset of $\mathbb{C}$ and $\{f_n\}$ is a sequence of holomorphic functions over Ω that uniformly converges to f over each compact subset $K \subset \Omega$. The limit function f is then holomorphic over Ω.

Moreover, the uniform convergence of $\{f_n\}$ to f implies the uniform convergence of $\{f_n'\}$ to f' over each compact set $K \subset \Omega$.

Proof. First, we prove that f is holomorphic by Morera's theorem (Theorem 15.1). For each point $z_0 \in \Omega$, we take a small disk $D_\varepsilon(z_0) \subset \Omega$. Then for each triangle T with $\overline{T} \subset D_\varepsilon(z_0)$, uniform convergence of $\{f_n\}$ to f implies that
$$\lim_{n\to\infty} \int_{\partial T} f_n(z)\,\mathrm{d}z = \int_{\partial T} \lim_{n\to\infty} f_n(z)\,\mathrm{d}z = \int_{\partial T} f(z)\,\mathrm{d}z = 0.$$
Here, we can swap the limit and the integral because ∂T is compact. Applying Morera's theorem, we can say that f is holomorphic over $D_\varepsilon(z_0)$ and thus also at z_0. As such, f is holomorphic over Ω as well.

Next, we will take an arbitrary compact subset $K \subset \Omega$, and we'll show that $\{f_n'\}$ uniformly converges to f' over K.

For each $z \in K$, suppose we let $\delta_z > 0$ be a radius such that $\overline{D_{2\delta_z}(z)} \subset \Omega$. Then, $\{D_{\delta_z(z)} \mid z \in K\}$ forms an open cover of K. Since K is compact, we can pick a finite subcover $\{D_{\delta_{z_i}}(z_i) \mid i = 1, \ldots, L\}$.

Let us define $r := \inf\left\{\delta_{z_i} \mid i = 1, \ldots, L\right\}$ as the smallest radius in the subcover. Since there are only finitely many such disks, $r > 0$.

For any $z \in K$, there must exist some i such that $z \in D_{\delta_{z_i}}(z_i)$; if we take any point $w \in D_r(z)$ (with the r defined earlier), then we can say
$$\begin{aligned} |w - z_i| &\le |w - z| + |z - z_i| && \text{(triangle inequality)} \\ &\le r + \delta_{z_i} \le 2\delta_{z_i} \end{aligned}$$

This means that $w \in D_{2\delta_{z_i}}(z_i) \subset \overline{D_{2\delta_{z_i}}(z_i)} \subset \Omega$.

Suppose we now construct $\tilde{K} = \bigcup_{z_i} \overline{D_{2\delta_{z_i}}(z_i)}$. Since $\tilde{K}$ is a union of compact subsets, it is also compact; we also know that $\tilde{K}$ contains K, since it contains a cover of K.

Hence, we've just shown that $D_r(z) \subset \tilde{K}$ for any $z \in K$. (Specifically, any $w \in D_r(z)$ also satisfies $w \in \overline{D_{2\delta_{z_i}}(z_i)}$, and we've defined $\tilde{K}$ to be the union of all such disks.)

Further, since $\{f_n\}$ uniformly converges to f over $\tilde{K}$ (as it is a subset of Ω), we know that there exists some $N > 0$ such that for any $n > N$ and $w \in \tilde{K}$ we have

$$\left|f_n(w) - f(w)\right| < \varepsilon.$$

Finally, for any $z \in K$, we can consider $D_r(z)$ (again with the r defined earlier) with Cauchy's inequality for the holomorphic functions $f_n - f$ (with derivative $f_n' - f'$):

$$\begin{aligned}\left|f_n'(z) - f'(z)\right| &\le \frac{1}{r} \sup_{w \in \partial D_r(z)} \left|f_n(w) - f(w)\right| \\ &\le \frac{1}{r} \sup_{w \in \tilde{K}} \left|f_n(w) - f(w)\right| \\ &\le \frac{\varepsilon}{r}\end{aligned}$$

This shows that $\{f_n' - f'\}$ uniformly converges to zero over K; that is, $\{f_n'\}$ uniformly converges to f' over K.

□

Corollary 15.4

An immediate corollary to Theorem 15.3 is that $\{f_n^{(k)}\}$ uniformly converges to $f^{(k)}$ over each compact subset K in Ω for any $k = 1, 2, \ldots$.

We can make a few remarks on the previous theorem. Firstly, we have a notion of *compact convergence*, which is less restrictive than uniform convergence on the whole domain, and makes some interesting examples survive.

Definition 15.5: Compact Convergence

Suppose $\{f_n\}$ is a sequence of complex functions on Ω. We say that $\{f_n\}$ *compactly converges* to f on Ω if $\{f_n\}$ converges uniformly to f over each compact subset K of Ω.

Example 15.6

Here's an example of how uniform convergence over the domain is more restrictive than compact convergence. If we work in $\Omega = \mathbb{C}$ and require that the sequence of holomorphic functions $\{f_n\}$ uniformly converges to some f, then $\{f_n - f\}$ must be a sequence of bounded entire functions that converges to zero.

By Liouville's theorem (Theorem 13.6), the only possible functions that satisfy this requirement are constant zero functions. This means that we lose all interesting examples that are preserved if we only assume compact convergence.

One such example is the sequence of functions

$$\{E_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!}\},$$

which is compactly convergent to e^z, but not uniformly convergent on $\mathbb{C}$.

Example 15.7

Another example for compact convergence is with the following sequence of functions

$$f_n(z) = z^n, \quad n = 1, 2, \ldots.$$

This sequence of (holomorphic) functions converges pointwise to the constant zero function on the unit disk $D_1(0)$, but not uniformly.

However, if we take any compact set $K \subset D_1(0)$, we can let

$$r := \sup\{|z| : z \in K\}.$$

Since the norm function $\|\cdot\| : K \to \mathbb{R}$ is a continuous function on the compact set K, its maxima can be achieved; this means that $0 \le r < 1$. Then, for any $z \in K$, we have

$$\left|z^n\right| = |z|^n < r^n.$$

Since $r^n \to 0$ as $n \to \infty$, we can see that $\{z^n\}$ uniformly converges to 0 on K, and hence f_n compactly converges to 0 on $D_1(0)$.

3/15/2022

Lecture 16

Holomorphic Functions in terms of integrals, Schwarz Reflection Principle

We know that the sum of finitely many holomorphic functions on some domain Ω is still a holomorphic function on Ω. We can extend this result to the continuous case with the following theorem.

Theorem 16.1

Suppose Ω is an open subset of $\mathbb{C}$, and let $g : \Omega \times [0,1] \to \mathbb{C}$ be a function in the continuous $[0,1]$-family of holomorphic functions defined on Ω. That is,

- g is continuous on $\Omega \times [0,1]$
- For each $s \in [0,1]$,

  $$g_s := g(\cdot, s) : \Omega \to \mathbb{C}$$

  is a holomorphic function

Then, the integral

$$f(z) := \int_0^1 g(z,s)\,\mathrm{d}s$$

is holomorphic over Ω.

Proof. Since g is continuous in s, the Riemann integral is well-defined and as such we can use a sequence of Riemann sums to approach the integral. That is, for each $n = 1, 2, \ldots$, we define

$$f_n(z) := \frac{1}{n}\sum_{k=1}^{n} g\left(z, \frac{k}{n}\right),$$

which is a holomorphic function over Ω.

Taking any $z \in \Omega$, we can use $\{f_n\}$ to prove that f is holomorphic at z. To do this, suppose we have a ball D such that $\overline{D} \subset \Omega$. We'll prove that $\{f_n\}$ converges to f uniformly on $\overline{D}$.

Notice that since g is continuous, g is uniformly continuous over $\overline{D} \times [0,1]$, since $\overline{D} \times [0,1]$ is compact. This means that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|s_1 - s_2| < \delta \implies \sup_{w \in \overline{D}} \left| g(w, s_1) - g(w, s_2) \right| < \varepsilon.$$

Now, if n is large enough such that $\frac{1}{n} < \delta$, then it follows that for any $w \in \overline{D}$,

$$\begin{aligned}
\left| f_n(w) - f(w) \right| &= \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} g\left(w, \frac{k}{n}\right) - g(w,s)\,\mathrm{d}s \right| \\
&\le \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left| g\left(w, \frac{k}{n}\right) - g(w,s) \right| \mathrm{d}s \\
&\le \sum_{k=1}^{n} \underbrace{\sup_{s \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} \left| g\left(w, \frac{k}{n}\right) - g(w,s) \right|}_{<\varepsilon} \cdot \underbrace{\left(\frac{k}{n} - \frac{k-1}{n} \right)}_{\frac{1}{n}} \\
&\le \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon
\end{aligned}$$

This shows that $\{f_n\}$ converges uniformly to f over $\overline{D}$. As such, for every compact subset K in D, $\{f_n\}$ converges uniformly to f over K. Using Theorem 15.3 for D, we obtain that f is holomorphic over D, and in particular, holomorphic at z. □

Proof. We can also prove this theorem with Morera's theorem (Theorem 15.1); for any triangle T such that $\overline{T} \subset \Omega$, we have

$$\begin{aligned}
\int_{\partial T} f(z)\,\mathrm{d}z &= \int_{\partial T} \int_0^1 g(z,s)\,\mathrm{d}s\,\mathrm{d}z && \text{(def. of } f\text{)} \\
&= \int_0^1 \int_{\partial T} g(z,s)\,\mathrm{d}z\,\mathrm{d}s && \text{(Fubini's theorem)} \\
&= \int_0^1 0\,\mathrm{d}s && \text{(Goursat's theorem)} \\
&= 0
\end{aligned}$$

Here, we utilized Fubini's theorem to swap the integrals, which can be derived from Fubini's theorem for real functions.

As such, by Morera's theorem, we can conclude that f must be holomorphic on Ω. □

16.1 Schwarz Reflection Principle

Now, we'll talk a little bit about the properties of functions on open sets Ω that are symmetric about the real axis.

Definition 16.2: Symmetric about the real axis

An open set Ω is *symmetric* about the real axis if $z \in \Omega \iff \overline{z} \in \Omega$.

Specifically, Fig. 16.1 illustrates an example of a symmetric open set Ω.

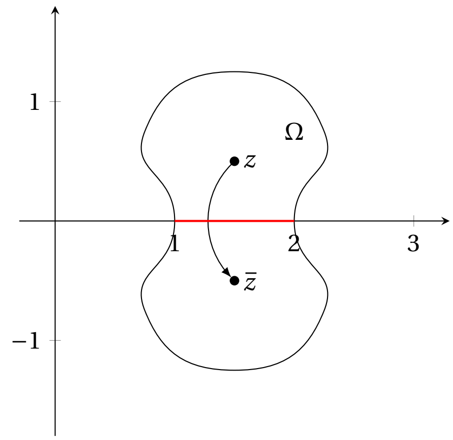


Figure 16.1: Example of an open set Ω symmetric about the real axis

Here, we define

$$\begin{aligned}\Omega^+ &= \{z \in \Omega \mid \operatorname{Im}(z) > 0\}\\ \Omega^- &= \{z \in \Omega \mid \operatorname{Im}(z) < 0\}\\ I &= \{z \in \Omega \mid \operatorname{Im}(z) = 0\}\end{aligned}$$

Note that $\Omega = \Omega^+ \sqcup \Omega^- \sqcup I$ (i.e. Ω is the disjoint union of the above sets). Further, notice that since Ω is open in $\mathbb{C}$, Ω^+ and Ω^- are both open in $\mathbb{C}$.

Suppose we have a holomorphic function f on Ω^+. One question we'll answer today is: when can f be holomorphically extended to Ω? This is answered with the Schwarz Reflection Principle.

Theorem 16.3: Schwarz Reflection Principle

Suppose f is holomorphic on Ω^+, and continuously extends to I. That is, there exists continuous function $\tilde{f} : \Omega^+ \sqcup I \to \mathbb{C}$ such that $\tilde{f}\big|_{\Omega^+} = f$.

Then, when $\tilde{f}\big|_I$ is a real function, f has a holomorphic extension to Ω. Further, such an extension must be unique by the unique continuation theorem (Theorem 14.5).

Proof. Suppose we define an F on $\Omega = \Omega^+ \sqcup \Omega^- \sqcup I$ such that

$$F(z) = \begin{cases} f(z) = \tilde{f}(z) & z \in \Omega^+ \\ \tilde{f}(z) & z \in I \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}.$$

Our task is to show that $F(z)$ is holomorphic on Ω. This means that $F(z)$ is our holomorphic extension from Ω^+ to Ω.

The first step is to check that $F(z)$ is continuous on Ω.

If $z_0 \in \Omega^+$, then there exists an open disk around z_0 such that $F(z) = f(z)$ in the open disk. Since f is continuous, F is also continuous at z_0.

If $z_0 \in \Omega^-$, then in an open disk (staying in Ω^-) around z_0, $F(z) = \overline{f(\bar{z})}$. Notice that F in this case is a compositing of three continuous maps

$$z \xmapsto{\text{conj.}} \bar{z} \xmapsto{f} f(\bar{z}) \xmapsto{\text{conj.}} \overline{f(\bar{z})}.$$

This means that F is continuous as well.

Lastly, if $z_0 \in I$, then it is enough to check that for any sequence $\{z_n\}$ in Ω, $z_n \to z_0 \implies F(z_n) \to F(z_0)$. Equivalently, we need to show that $|F(z_n) - F(z_0)| \to 0$; noticing that $F(z_0) = \tilde{f}(z_0) \in \mathbb{R}$, we have

$$|F(z_n) - F(z_0)| = \begin{cases} \left|\tilde{f}(z_n) - \tilde{f}(z_0)\right| & z_n \in \Omega^+ \sqcup I \\ \left|\overline{f(\overline{z}_n)} - \tilde{f}(z_0)\right| & z_n \in \Omega^- \end{cases}.$$

We can now looking at each of these two cases individually. If $z_n \in \Omega^+ \sqcup I$, then we know that $\left|\tilde{f}(z_n) - \tilde{f}(z_0)\right| \to 0$ as $n \to \infty$, since $\tilde{f}$ is assumed to be continuous on $\Omega^+ \sqcup I$.

If $z_n \in \Omega^-$, then we have

$$\begin{aligned} \left|\overline{f(\overline{z}_n} - \tilde{f}(z_0)\right| &= \left|\overline{\tilde{f}(\overline{z}_n)} - \tilde{f}(z_0)\right| && \text{(since } \overline{z}_n \in \Omega^+\text{)} \\ &= \left|\overline{\overline{\tilde{f}(\overline{z}_n)} - \tilde{f}(z_0)}\right| && \text{(since } \left|\overline{z}\right| = |z|\text{)} \\ &= \left|\tilde{f}(\overline{z}_n) - \tilde{f}(z_0)\right| && \text{(since } \tilde{f}(z_0) \in \mathbb{R}\text{)} \end{aligned}$$

Further, since $z_n \to z_0 \iff \overline{z}_n \to z_0$ (since $z_0 \in \mathbb{R}$), and since we know that $\tilde{f}$ is continuous, this quantity also goes to 0 as $n \to \infty$.

Together, these two parts show that F is continuous at z_0.

The next step is to show that F is holomorphic on Ω. Again, we can split this up into cases.

If $z_0 \in \Omega^+$, then $F(z) = f(z)$ near z_0, and we know that f is holomorphic. This means that F is also holomorphic at z_0.

If $z_0 \in \Omega^-$, then let us take a disk $D_\varepsilon(z_0) \subseteq \Omega^-$; this means that $D_\varepsilon(\overline{z}_0) \subseteq \Omega^-$ by symmetry.

We know that f is holomorphic on Ω^+, so on $D_\varepsilon(\overline{z}_0)$, we can write f as a power series centered at $\overline{z}_0$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \overline{z}_0)^n,$$

for some $a_n \in \mathbb{C}$. Then, for any $w \in D_\varepsilon(z_0)$, we have $\overline{w} \in D_\varepsilon(\overline{z}_0)$, and we can write

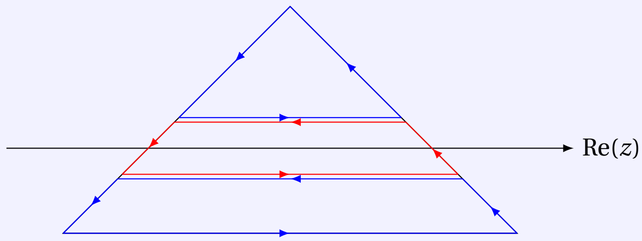
$$\begin{aligned} F(w) &= \overline{f(\overline{w})} \\ &= \overline{\sum_{n=0}^{\infty} a_n (\overline{w} - \overline{z}_0)^n} \\ &= \sum_{n=0}^{\infty} \overline{a_n (\overline{w} - \overline{z}_0)^n} && \text{(since } \overline{a+b} = \overline{a} + \overline{b}\text{)} \\ &= \sum_{n=0}^{\infty} \overline{a}_n \left(\overline{\overline{w} - \overline{z}_0}\right)^n && \text{(since } \overline{ab} = \overline{a}\overline{b}\text{)} \\ &= \sum_{n=0}^{\infty} \overline{a}_n (w - z_0)^n \end{aligned}$$

This is a convergent power series in w (since it has the same coefficients as the series for f), so F is holomorphic on $D_\varepsilon(z_0)$, and in particular at z_0.

(There are other methods to prove this case as well, like with the definition of holomorphicity, Morera's theorem, etc.)

Lastly, we need to show that F is holomorphic at $z_0 \in I$. Suppose we take an open disk D in Ω centered at z_0; we can show that F is holomorphic on D through Morera's theorem, and therefore also holomorphic at z_0.

To do this, let us consider a triangle T with $\overline{T} \subseteq D$; we need to show that $\int_{\partial T} F(z)\,\mathrm{d}z = 0$.



We can split the triangle T into three parts for some $\varepsilon > 0$:

- an upper triangle $T_\varepsilon^+ = \{z \in T \mid \operatorname{Im}(z) \geq \varepsilon\}$
- a lower triangle $T_\varepsilon^- = \{z \in T \mid \operatorname{Im}(z) \leq -\varepsilon\}$
- a middle trapezoid $T_\varepsilon = \{z \in T \mid -\varepsilon \leq \operatorname{Im}(z) \leq \varepsilon\}$

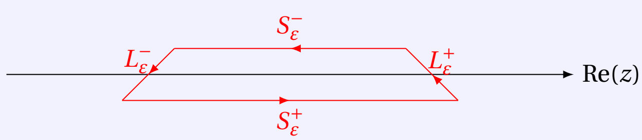
We can then split up the integral as

$$\begin{aligned}\int_{\partial T} F(z)\,\mathrm{d}z &= \underbrace{\int_{\partial T_\varepsilon^+} F(z)\,\mathrm{d}z}_{0} + \underbrace{\int_{\partial T_\varepsilon^-} F(z)\,\mathrm{d}z}_{0} + \int_{\partial T_\varepsilon} F(z)\,\mathrm{d}z \\ &= \int_{\partial T_\varepsilon} F(z)\,\mathrm{d}z\end{aligned}$$

Here, since T_ε^+ and T_ε^- lie entirely in Ω^+ and Ω^- respectively, and we know that F is holomorphic in both regions, the first two terms are zero by Goursat's theorem (equivalently by Cauchy's theorem). This means that we just need to show that

$$\lim_{\varepsilon\to 0^+} \int_{\partial T_\varepsilon} F(z)\,\mathrm{d}z = 0.$$

We can split up the trapezoid into four line segments such that $\partial T_\varepsilon = L_\varepsilon^+ \cup L_\varepsilon^- \cup S_\varepsilon^+ \cup S_\varepsilon^-$.



First looking at the horizontal line segments S_ε^+ and S_ε^-, we can see that as $\varepsilon \to 0$, the integral along these two segments cancel each other out, since they have opposite orientation. This means that

$$\lim_{\varepsilon\to 0^+} \int_{S_\varepsilon^+} F(z)\,\mathrm{d}z + \int_{S_\varepsilon^-} F(z)\,\mathrm{d}z = \int_{S_0^+} F(z)\,\mathrm{d}z + \int_{S_0^-} F(z)\,\mathrm{d}z = 0.$$

Next, we can look at the slanted segments crossing the real axis. If we look at the length of these segments, we have

$$\left|\int_{L_\varepsilon^+} F(z)\,\mathrm{d}z\right| \leq \underbrace{\sup_{z\in L_\varepsilon^+} |F(z)|}_{\text{bounded}} \cdot \underbrace{\operatorname{length}(L_\varepsilon^+)}_{\to 0}.$$

Here, we know that $|F(z)|$ is bounded above along L_ε^+ because F is continuous; further, the length of L_ε^+ goes to zero as $\varepsilon \to 0^+$, so their product also goes to zero as $\varepsilon \to 0^+$.

The same reasoning holds for L_ε^-, so we also have $\left|\int_{L_\varepsilon^-} F(z)\,\mathrm{d}z\right| \to 0$ as $\varepsilon \to 0^+$.

Putting all of this together, we have

$$\left|\int_{\partial T_\varepsilon} F(z)\,\mathrm{d}z\right| \le \left|\int_{L_\varepsilon^+} F(z)\,\mathrm{d}z\right| + \left|\int_{L_\varepsilon^-} F(z)\,\mathrm{d}z\right| + \left|\int_{S_\varepsilon^+} F(z)\,\mathrm{d}z\right| + \left|\int_{S_\varepsilon^-} F(z)\,\mathrm{d}z\right|.$$

The RHS goes to zero as $\varepsilon \to 0^+$, so we have

$$\int_{\partial T} F(z)\,\mathrm{d}z = 0.$$

By Morera's theorem, we can finally conclude that F is holomorphic on D, and thus also at z_0. □

3/17/2022

Lecture 17

Symmetry Principle, Singularities and Zeroes

17.1 Symmetry Principle

We can extend the Schwarz reflection principle with the following theorem:

Theorem 17.1: Symmetry Principle

Let $\Omega \subseteq \mathbb{C}$ be an open set, and suppose Ω is divided by some piecewise-smooth curve I into two open sets Ω^+ and Ω^-. Suppose we have two functions f^+ and f^- that are holomorphic on Ω^+ and Ω^- respectively. Further suppose that both can be extended to I continuously (we will let f^+ and f^- denote the extensions), and let $f^+(z) = f^-(z)$ for every $z \in I$.

Then, the function

$$F(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(z) = f^-(z) & z \in I \\ f^-(z) & z \in \Omega^- \end{cases}$$

is holomorphic on Ω.

Proof. We only need to check that for every $z_0 \in I$, $F(z)$ is holomorphic at z_0. To do this, we can actually use the exact same method as we did for the proof of the Schwarz reflection principle (Theorem 16.3). □

One important remark to make here is that this means that for the two holomorphic functions f^+ on Ω^+ and f^- on Ω^-, a continuous gluing of f^+ and f^- is automatically a holomorphic gluing.

Note that this is *not* the case for reals and smoothness; for smooth functions f^+ on Ω^+ and f^- on Ω, a continuous gluing *does not* imply a smooth gluing.

Example 17.2

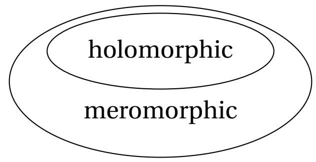
As an example, suppose we take

$$\begin{aligned} f^+(x,y) &= x && \text{over } \Omega^+ = \mathbb{R}^+ \times \mathbb{R} \\ f^-(x,y) &= -x && \text{over } \Omega^- = \mathbb{R}^- \times \mathbb{R} \end{aligned}$$

With $I = \{0\} \times \mathbb{R}$, f^+ and f^- can continuously extend to I by 0, but this extension is only continuous; it is not smooth.

17.2 Singularities

The next big topic we'll be covering is the notion of *meromorphic* functions, a superset of holomorphic functions.



For holomorphic functions on a set Ω, it must be the case that every single point in Ω is a holomorphic point (i.e. the function is holomorphic at every point in Ω). However, for meromorphic functions, we allow *singularities* in Ω, where the function is not defined or is not holomorphic. More specifically, we don't want these singularities to be too "bad"; they're called *pole singularities*.

We'll now define these concepts in a more rigorous fashion.

Definition 17.3: Isolated singularity

A point $z_0 \in \mathbb{C}$ is an *isolated singularity* of a function f if there is a disk D around z_0 such that f is not defined on z_0, but f is holomorphic on $D \setminus \{z_0\}$.

Example 17.4

Consider the function

$$f(z) = \frac{z^2}{z}.$$

We can see that f is undefined at $z_0 = 0$, but f is holomorphic everywhere else; it's the same as $f(z) = z$ if $z \neq 0$.

This means that $z_0 = 0$ is an isolated singularity.

It should be clear that this singularity isn't too bad; it's *removable*. That is, we can define

$$\tilde{f}(z) = \begin{cases} f(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

which is holomorphic everywhere.

Definition 17.5: Removable singularity

An isolated singularity $z_0 \in \mathbb{C}$ of a function f is a *removable singularity* if there exists a holomorphic extension of f to z_0.

Specifically z_0 is a removable singularity if $\lim_{z\to z_0} f(z)$ exists, but f is not defined at z_0.

Example 17.6

Consider the function

$$f(z) = \frac{1}{z}.$$

We can see that f is undefined as $z_0 = 0$, but f is holomorphic on $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$. This means that $z_0 = 0$ is an isolated singularity.

However, f has no holomorphic extension to 0, so 0 is *not* a removable singularity.

Put another way, $\lim_{z\to 0} f(z)$ DNE, so 0 is not a removable singularity. We also have $\lim_{z\to 0}|f(z)| = +\infty$, which means that we classify $z_0 = 0$ as a *pole singularity* (defined later).

Example 17.7

Consider the function

$$f(z) = e^{\frac{1}{z}}.$$

We can see that $z_0 = 0$ is an isolated singularity, since $f(z)$ is holomorphic on $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$.

The limit $\lim_{z\to 0} f(z)$ DNE, so $z_0 = 0$ is not removable. Looking at the limit of the norm, we have

$$\lim_{z\to 0}|f(z)| = \lim_{z\to 0}\left|e^{\frac{1}{z}}\right|$$

If we let $z = x + iy$, we can see that $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{1}{|z|^2}\cdot(x - iy)$; plugging this in, we have

$$\begin{aligned} &= \lim_{z\to 0}\left|e^{\frac{x}{|z|^2} - i\frac{y}{|z|^2}}\right| \\ &= \lim_{z\to 0} e^{\frac{x}{|z|^2}} \end{aligned}$$

On the real axis, the exponent becomes $\frac{x}{|z|^2} = \frac{x}{x^2} = \frac{1}{x}$.

If we take the limit as $x \to 0^+$, then $\frac{1}{x} \to +\infty$ and $e^{\frac{1}{x}} \to +\infty$.

If we take the limit as $x \to 0^-$, then $\frac{1}{x} \to -\infty$ and $e^{\frac{1}{x}} \to 0$.

Since these limits don't match up, we know that $\lim_{z\to 0}|f(z)|$ DNE. We call this an *essential singularity*.

Definition 17.8: Essential singularity

An isolated singularity $z_0 \in \mathbb{C}$ of a function f is an *essential singularity* if

$$\lim_{z\to z_0}|f(z)|$$

does not exist.

We'll be looking into *pole singularities* in this section. We can roughly define a pole singularity z_0 of a function if it is a *zero* of the reciprocal $\frac{1}{f}$. To give a formal definition, we'll first look at zeroes of holomorphic functions.

Definition 17.9: Zero point

Suppose f is a holomorphic function on an open set $\Omega \subseteq \mathbb{C}$. A point $z_0 \in \Omega$ is a *zero point* of f if $f(z_0) = 0$.

Definition 17.10: Isolated zero point

Suppose f is a holomorphic function on an open set $\Omega \in \mathbb{C}$. A point $z_0 \in \Omega$ is an *isolated zero point* of f if there is some disk D around z_0 such that z_0 is the only zero point of f in D.

An important note is that the unique continuation theorem allows us to conclude that whenever f is not the constant zero function, then every zero must be isolated.

Theorem 17.11: Zeroes of holomorphic functions are isolated

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, and suppose f is a nonzero holomorphic function on Ω. Then, for each zero point $z_0 \in \Omega$ of f, there exists some open disk D centered at z_0 such that

$$f(z) = (z - z_0)^n g(z),$$

with $g(z)$ holomorphic and nowhere vanishing on D, and with $n \in \mathbb{Z}^+$.

Proof. Since f is holomorphic on Ω, it is analytic and there is some disk D around z_0 such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

on D.

Since z_0 is an isolated zero, we know that $f(z_0) = 0 = a_0$. This means that

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

Since we're assuming that f is not constant zero, there must exist some k such that $a_k \neq 0$; this forces $a_1 = a_2 = \cdots = a_{k-1} = 0$. As such, we can write

$$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots$$

with $a_k \neq 0$. Factoring out $(z - z_0)^k$, we have

$$= (z - z_0)^k \underbrace{\left(a_k + a_{k+1}(z - z_0) + a_{k+2}(z - z_0)^2 + \cdots\right)}_{g(z)}$$

Here, $g(z)$ is analytic with $g(z_0) = a_k \neq 0$. We can also shrink D to a smaller disk such that $g(z)$ is nowhere vanishing on D. □

Definition 17.12: Order of a zero

The *order* or *degree* of a zero $z_0 \in \mathbb{C}$ of a function f is the value of n when we rewrite

$$f(z) = (z - z_0)^n g(z).$$

Lemma 17.13

The order of any zero is uniquely determined by z_0.

Proof. Suppose we have

$$f(z) = (z - z_0)^m g_1(z) = (z - z_0)^n g_2(z)$$

such that g_1 and g_2 are nowhere vanishing. Suppose for contradiction that $m \neq n$.

WLOG, suppose $m > n$; it then follows that $(z - z_0)^{m-n} g_1(z) = g_2(z)$. Taking the limit as $z \to z_0$, the LHS goes to zero, so the RHS is forced to tend toward zero as well.

However, this means that $g_2(z_0) = 0$, since g_2 is holomorphic—this is a contradiction, since we assumed that g_2 was nowhere vanishing.

As such, it must have been the case that $m = n$, and the order of a zero is uniquely determined by z_0. □

Definition 17.14: Simple zero

A *simple zero* is a zero of order $n = 1$.

Example 17.15

Consider the function

$$f(z) = (z - z_1)^{n_1}(z - z_1)^{n_2} \cdots (z - z_k)^{n_k}.$$

Here, $z_1, \ldots, z_k$ are the distinct roots of $f(z)$, and n_j is the order of the zero z_j.

For example, if

$$f(z) = (z-1)^2(z+i)^3 z^4,$$

then we have zeroes $z_1 = 1$, $z_2 = -i$, and $z_3 = 0$, with orders $n_1 = 2$, $n_2 = 3$, and $n_3 = 4$ respectively.

Example 17.16

Consider the function

$$f(z) = \sin(z).$$

We can see that $z_0 = 0$ is an isolated zero. What is the order of this zero? We need to find an exponent n and a function $g(z)$ such that

$$f(z) = (z-0)^n g(z).$$

We know the series expansion of $\sin(z)$ is

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ &= z \underbrace{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)}_{g(z)} \end{aligned}$$

This means that $\sin z$ has 0 as a simple zero.

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Lecture 18

Pole Singularities, Residue

Definition 18.1: Pole Singularity

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with $z_0 \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{z_0\}$.

z_0 is a *pole singularity* of f if there exists an open disk D centered at z_0 such that $f(z) \neq 0$ for all $z \in D^* = D \setminus \{z_0\}$,

$$g(z) = \begin{cases} \frac{1}{f(z)} & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

is holomorphic. As a result, $g(z)$ must have z_0 as an isolated zero.

Definition 18.2: Order of a pole

The *order* of a pole z_0 for a function $f(z)$ is the order of the corresponding zero for $g(z)$.

Definition 18.3: Simple pole

A *simple pole* is an order 1 pole.

Example 18.4

Consider the function

$$f(z) = \frac{1}{z^2}.$$

Here, let $\Omega = \mathbb{C}$, and $z_0 = 0$.

We can see that

$$g(z) = \begin{cases} \frac{1}{f(z)} = z^2 & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

This is equivalent to just $g(z) = z^2$, which is holomorphic on $\mathbb{C}$.

Since $z_0 = 0$ is an order 2 zero of $g(z)$, we can conclude that $z_0 = 0$ is an order 2 pole for $f(z) = \frac{1}{z^2}$.

Theorem 18.5

Suppose f is holomorphic on $\Omega^* = \Omega \setminus \{z_0\}$, with z_0 as a pole singularity of order n. Then, in a small open disk D around z_0, we can write f on $D^* = D \setminus \{z_0\}$ as

$$f(z) = \frac{h(z)}{(z - z_0)^n},$$

where $h(z)$ is nowhere vanishing on D.

Proof. Suppose we have a small disk D around z_0 in Ω. We know that

$$g(z) = \begin{cases} \frac{1}{f(z)} & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

is holomorphic, and that g has z_0 as an order n zero.

As such, we can write $g(z) = (z - z_0)^n \hat{h}(z)$, where $\hat{h}(z)$ is holomorphic and nowhere vanishing.

For $z \neq z_0$, we can write

$$f(z) = \frac{1}{g(z)} = \frac{1}{\hat{h}(z)(z - z_0)^n} = \frac{h(z)}{(z - z_0)^n},$$

where we define $h(z) = \frac{1}{\hat{h}(z)}$, which is nowhere vanishing on D and holomorphic. □

Example 18.6

From last time, we know that z and $\sin z$ have simple zeroes at $z_0 = 0$, so $f(z) = \frac{1}{z}$ and $f(z) = \frac{1}{\sin z}$ have simple poles at $z_0 = 0$.

Further, we can consider $f(z) = \frac{1}{e^z-1}$; we can expand $e^z - 1$ as

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ e^z - 1 &= z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ &= z \underbrace{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}_{g(z)} \end{aligned}$$

As such, $e^z - 1$ has a simple zero at $z_0 = 0$, and thus f has a simple pole at $z_0 = 0$.

Example 18.7

Consider the function

$$f(z) = \frac{1}{(z-1)^2(z-i)^3}.$$

We can see that f has singularities where $(z-1)^2(z-i)^3 = 0$, i.e. when $z_1 = 1$ and $z_2 = i$.

For $z_1 = 1$, we can write

$$f(z) = \frac{\frac{1}{(z-i)^3}}{(z-1)^2}.$$

The numerator is holomorphic and nowhere vanishing near $z_1 = 1$, so $f(z)$ has $z_1 = 1$ as an order 2 pole.

For $z_2 = i$, we can write

$$f(z) = \frac{\frac{1}{(z-1)^2}}{(z-i)^3}.$$

The numerator is holomorphic and nowhere vanishing near $z_2 = i$, so $f(z)$ has $z_2 = i$ as an order 3 pole.

One remark to make is if we have functions of the form

$$f(z) = \frac{g(z)}{p(z)},$$

where g is holomorphic and does not vanish at $z_1, z_2, \dots, z_k$, and p is a polynomial function of order n. In this case, we can write

$$p(z) = (z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_k)^{n_k},$$

with $z_1, z_2, \dots, z_k$ distinct, and $n_1 + n_2 + \cdots + n_k = n$.

If we have a function f of this form, then we can say that z_1 is an order n_1 pole, etc. and z_k is an order n_k pole.

Example 18.8

Consider the function

$$f(z) = \frac{z^2}{e^z - 1}.$$

We can see that $z_0 = 0$ is a zero for $e^z - 1$ and also a zero for z^2.

We can simplify f for $z \neq 0$ near $z_0 = 0$ as

$$f(z) = \frac{z^2}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots} = \frac{z}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}.$$

As $z \to 0$, we can see that $f(z) \to 0$, since the numerator approaches zero but the denominator approaches 1.

This means that f can be holomorphically extended to 0, and $z_0 = 0$ is a removable singularity, not a pole.

Example 18.9

Consider the function
$$f(z) = \frac{z}{(e^z - 1)^2}.$$
We can see that $z_0 = 0$ is a zero for $e^z - 1$, and we can rewrite $e^z - 1 = zg(z)$. This means that $(e^z - 1)^2 = z^2 g(z)^2$. Near 0, we can rewrite f as
$$f(z) = \frac{z}{z^2 g(z)^2} = \frac{1}{zg(z)^2} = \frac{\frac{1}{g(z)^2}}{z}.$$
Since $g(z)^2$ is nowhere vanishing, the numerator is nowhere vanishing and this means that $z_0 = 0$ is a simple pole.

Theorem 18.10

If z_0 is an order n pole of $f(z)$, then on a small disk D centered at z_0, we can write
$$f(z) = \frac{c_{-n}}{(z - z_0)^n} + \frac{c_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{c_{-1}}{z - z_0} + h(z),$$
where h is holomorphic and $c_{-n}, c_{-(n-1)}, \ldots, c_{-1} \in$, with $c_{-n} \neq 0$.

Proof. Consider a disk D around z_0; we can write
$$f(z) = \frac{g(z)}{(z - z_0)^n},$$
where $g(z)$ is nowhere vanishing and holomorphic on D.

Further, since g is analytic, we can write
$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$
noting that $g(z_0) \neq 0 \implies a_0 \neq 0$. Writing out $f(z)$ in full, we have
$$\begin{aligned} f(z) &= \frac{g(z)}{(z - z_0)^n} \\ &= \frac{a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots + a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots}{(z - z_0)^n} \\ &= \underbrace{\frac{a_0}{(z - z_0)^n} + \frac{a_1}{(z - z_0)^{n-1}} + \cdots + \frac{a_{n-1}}{(z - z_0)^1}}_{\text{principal part of } f(z)} + \underbrace{a_n + a_{n+1}(z - z_0) + \cdots}_{\text{holomorphic function } h(z)} \end{aligned}$$
□

Definition 18.11: Residue

The *residue* of f at a pole z_0 is the value of $c_{-1} \in$ in the expansion
$$f(z) = \frac{c_{-n}}{(z - z_0)^n} + \frac{c_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{c_{-1}}{z - z_0} + h(z)$$

on a small disk D around z_0.

As notation, we have

$$\operatorname{Res}_{z=z_0}(f) = c_{-1}.$$

This can be abbreviated as $\operatorname{Res}_{z_0}(f)$.

Theorem 18.12: Residue Formula (special case)

Suppose we have a closed loop γ around a pole $z_0 \in$ of a function f. Then, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i c_{-1} \iff c_{-1} = \frac{1}{2\pi i}\oint_\gamma f(z)\,\mathrm{d}z.$$

Proof. We can use the decomposition from Theorem 18.10 to write

$$\begin{aligned}
\oint_\gamma f(z)\,\mathrm{d}z &= \oint_\gamma \left(\frac{c_{-n}}{(z-z_0)^n} + \cdots + \frac{c_{-1}}{z-z_0} + h(z)\right)\mathrm{d}z \\
&= \oint_\gamma \frac{c_{-1}}{z-z_0}\,\mathrm{d}z \\
&= 2\pi i c_{-1}
\end{aligned}$$

Here, the second equality is due to the fact that every term of the form $\frac{c_{-k}}{(z-z_0)^k}$ has a primitive, except for when $k=-1$. This means that the contour integral of every other term over γ goes to zero, and we're left with just $\frac{c_{-1}}{z-z_0}$. □

How does one calculate the residue at a pole? One way is to use the decomposition in Theorem 18.10, and take C_{-1} directly.

Here is another way to calculate the residue.

Theorem 18.13

Suppose $f(z)$ has an order n pole z_0. Then,

$$\operatorname{Res}_{z=z_0}(f) = \frac{1}{(n-1)!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[(z-z_0)^n f(z)\right]\Big|_{z=z_0}.$$

Proof. We can write f as

$$\begin{aligned}
f(z) &= \frac{c_{-n}}{(z-z_0)^n} + \cdots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)^1} + h(z) \\
(z-z_0)^n f(z) &= c_{-n} + c_{-(n-1)}(z-z_0) + \cdots + c_{-1}(z-z_0)^{n-1} + h(z)(z-z_0)^n \\
\frac{\mathrm{d}}{\mathrm{d}z}\left((z-z_0)^n f(z)\right) &= c_{-(n-1)} + 2c_{-(n-2)}(z-z_0) + \cdots + (n-1)c_{-1}(z-z_0)^{n-2} + \frac{\mathrm{d}}{\mathrm{d}z}\left(h(z)(z-z_0)^n\right) \\
&\vdots \\
\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left((z-z_0)^n f(z)\right) &= (n-1)!c_{-1} + \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left(h(z)(z-z_0)^n\right) \\
&= (n-1)!c_{-1} + (z-z_0)(\cdots)
\end{aligned}$$

Here, in taking the $(n-1)$th derivative of $h(z)(z-z_0)^n$, we'll always have some factor of $(z-z_0)$ that survives, since the exponent is less than the order of the derivative.

Evaluated at $z = z_0$, then the second term disappears, leaving only

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\big((z-z_0)^n f(z)\big)\bigg|_{z=z_0} = (n-1)!c_{-1} \implies c_{-1} = \frac{1}{(n-1)!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\big((z-z_0)^n f(z)\big)\bigg|_{z=z_0}.$$

□

Example 18.14

Consider the function

$$f(z) = \frac{e^{-z}}{1+z^2}.$$

Find all singularities of $f(z)$ on $\mathbb{C}$, determine the types of all singularities, and calculate the residue at the poles.

We can see that f has singularities when $1+z^2 = 0$, i.e. at $z = \pm i$. Further, $e^{-z} \neq 0$ at $\pm i$, so $\pm i$ are simple pole singularities of f; we can rewrite it as

$$f(z) = \frac{e^{-z}}{(z+i)(z-i)}.$$

To compute the residue at these poles, we'll use the formula from Theorem 18.13.

We know that $z_1 = -i$ is a simple pole, so we want the $n-1 = 1-1 = 0$th derivative; this means we just need to evaluate at $z = -i$:

$$\begin{aligned}
\mathrm{Res}_{z=-i}(f) &= \frac{1}{(1-1)!}\big((z+i)^1 f(z)\big)\Big|_{z=-i} \\
&= \left((z+i)\frac{e^{-z}}{(z+i)(z-i)}\right)\bigg|_{z=-i} \\
&= \frac{e^{-z}}{z-i}\bigg|_{z=-i} \\
&= \frac{e^{i}}{-2i} = \frac{ie^{i}}{2}
\end{aligned}$$

Similarly, we know that $z_2 = i$ is a simple pole, so we can just evaluate at $z = i$ as well:

$$\begin{aligned}
\mathrm{Res}_{z=i}(f) &= \frac{1}{(1-1)!}\big((z-i)^1 f(z)\big)\Big|_{z=i} \\
&= \left((z-i)\frac{e^{-z}}{(z+i)(z-i)}\right)\bigg|_{z=i} \\
&= \frac{e^{-z}}{z+i}\bigg|_{z=i} \\
&= \frac{e^{-i}}{2i} = -\frac{ie^{-i}}{2}
\end{aligned}$$

Example 18.15

Consider the function

$$f(z) = \frac{(z^2-1)^2}{z^2(z-2)(2z-1)}.$$

Find all singularities of $f(z)$ on $\mathbb{C}$, determine the types of all singularities, and calculate the residue at the poles.

We can see that f has singularities whenever $z^2(z-2)(2z-1) = 0$, or at $z_1 = 0$, $z_2 = 2$, and $z_3 = \frac{1}{2}$.

The numerator doesn't vanish at any of these values, so these are all poles. Specifically $z_1 = 0$ is an order 2 pole, and $z_2 = 2$, $z_3 = \frac{1}{2}$ are both simple poles.

Now, let's look at the residues. If we just directly apply the formula, we have

$$\begin{aligned}
\operatorname{Res}_{z=0}(f) &= \frac{1}{(2-1)!}\,\frac{\mathrm{d}}{\mathrm{d}z}\Big((z-0)^2 f(z)\Big)\Big|_{z=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\cdot\frac{(z^2-1)^2}{z^2(z-2)(2z-1)}\right)\bigg|_{z=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(z^2-1)^2}{(z-2)(2z-1)}\right)\bigg|_{z=0} \\
&= \frac{5}{4}
\end{aligned}$$

We can also calculate this same residue using the definition (i.e. the expansion), though we'll first do some simplifications to make our lives easier.

If we expand the numerator, we have

$$\begin{aligned}
\frac{(z^2-1)^2}{z^2(z-2)(2z-1)} &= \frac{z^4-2z^2+1}{z^2(z-2)(2z-1)} \\
&= \frac{z^4}{z^2(z-2)(2z-1)} - \frac{2z^2}{z^2(z-2)(2z-1)} + \frac{1}{z^2(z-2)(2z-1)} \\
&= \frac{z^2}{(z-2)(2z-1)} - \frac{2}{(z-2)(2z-1)} + \frac{1}{z^2(z-2)(2z-1)}
\end{aligned}$$

Notice that the first two terms are holomorphic near $z_1 = 0$, so they do not contribute toward the residue (namely, they'd get bundled in to the remaining $h(z)$ function not in the principal part of f.

This means that we just need to calculate

$$\operatorname{Res}_{z=0}(f) = \operatorname{Res}_{z=0}\left(\frac{1}{z^2(z-2)(2z-1)}\right).$$

(Note that this simplification could have been used with the formulas well.)

We can then use the geometric series expansion here to rewrite

$$\begin{aligned}
\frac{1}{z-2} &= -\frac{1}{2-z} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right) \\
\frac{1}{2z-1} &= -\frac{1}{1-2z} = -\big(1+2z+(2z)^2+\cdots\big)
\end{aligned}$$

This means that we have

$$\begin{aligned}
\frac{1}{z^2(z-2)(2z-1)} &= \frac{1}{z^2}\cdot -\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)\cdot -\big(1+2z+(2z)^2+\cdots\big) \\
&= \frac{1}{2z^2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)\big(1+2z+(2z)^2+\cdots\big)
\end{aligned}$$

Since $z_1 = 0$ is an order 2 pole, the residue is the coefficient for the $\frac{1}{z}$ term. To get this term, we'd need to look at the coefficients for the resulting order 1 term from the product of the geometric series.

This comes from $1\cdot 2z + \frac{z}{2}\cdot 1 = \frac{5}{2}z$ in the product of the geometric series, giving us a final term of $\frac{1}{2z^2}\cdot\frac{5}{2}z = \frac{5}{4z}$.

This means that the residue is $\operatorname{Res}_{z=0}(f) = \frac{5}{4}$; this matches what we had before.

$$\begin{aligned}
\operatorname{Res}_{z=2}(f) &= \left((z-2)\cdot\frac{(z^2-1)^2}{z^2(z-2)(2z-1)}\right)\bigg|_{z=2} \\
&= \left(\frac{(z^2-1)^2}{z^2(2z-1)}\right)\bigg|_{z=2} = \frac{9}{12} = \frac{3}{4} \\
\operatorname{Res}_{z=\frac{1}{2}}(f) &= \left(\left(z-\frac{1}{2}\right)\cdot\frac{(z^2-1)^2}{z^2(z-2)(2z-1)}\right)\bigg|_{z=\frac{1}{2}}
\end{aligned}$$

Here, to make the denominator conform to the $(z-\frac{1}{2})$ term, we can divide by 2:

$$\begin{aligned}
&= \left(\frac{(z^2-1)^2}{2z^2(z-2)}\right)\bigg|_{z=\frac{1}{2}} \\
&= \frac{\frac{9}{16}}{\frac{1}{2}\cdot\left(\frac{3}{2}\right)} = \frac{9}{16}\left(-\frac{4}{3}\right) = -\frac{3}{4}
\end{aligned}$$

To summarize, we have

$$\operatorname{Res}_0(f) = \frac{5}{4} \qquad \operatorname{Res}_2(f) = \frac{3}{4} \qquad \operatorname{Res}_{\frac{1}{2}}(f) = -\frac{3}{4}$$

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Lecture 19

Residue Formula

Recall from last time that we covered a special case of the residue formula:

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}_{z_0}(f) \iff \operatorname{Res}_{z_0}(f) = \frac{1}{2\pi i}\oint_\gamma f(z)\,\mathrm{d}z.$$

Today we'll look at the general form of the residue formula.

Theorem 19.1: Residue Formula

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with f holomorphic on $\Omega \setminus \{z_1, z_2, \dots, z_k\}$, where $z_1, z_2, \dots, z_k$ are distinct pole singularities of f.

For any open set R with ∂R piecewise smooth that includes the poles $z_1, z_2, \dots, z_k$, and $\overline{R} \subseteq \Omega$, we have

$$\oint_{\partial R} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z_j}(f).$$

Proof. Let $\varepsilon > 0$ be small enough such that the disks $D_\varepsilon(z_j)$ centered at each pole don't intersect.

If we look at the boundary $\partial\left(R \setminus \bigcup_{j=1}^k D_\varepsilon(z_j)\right)$, the integral over this boundary is equivalent to taking the integral over $\partial R \cup \bigcup_{j=1}^k \bar{\partial} D_\varepsilon(z_j)$, where $\bar{\partial}$ is the boundary with reverse orientation.

We know that f is holomorphic on $R \setminus \bigcup_{j=1}^k D_\varepsilon(z_j)$, so by Cauchy's theorem,

$$\begin{aligned}
0 &= \int_{\partial R \cup \bigcup_{j=1}^k \bar{\partial} D_\varepsilon(z_j)} f(z)\,\mathrm{d}z \\
&= \oint_{\partial R} f(z)\,\mathrm{d}z - \sum_{j=1}^{k} \oint_{\partial D_\varepsilon(z_j)} f(z)\,\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}\oint_{\partial R} f(z)\,\mathrm{d}z &= \sum_{j=1}^{k} \oint_{\partial D_\varepsilon(z_j)} f(z)\,\mathrm{d}z \\ &= \sum_{j=1}^{k} 2\pi i \operatorname{Res}_{z_j}(f)\end{aligned}$$

Here, in the last equality, we use the fact that in each $D_\varepsilon(z_j)$, there is only one pole singularity z_j, and we can use the prior special case of the residue formula to simplify.

Factoring out the $2\pi i$, we get our desired result

$$\oint_{\partial R} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z_j}(f).$$

□

19.1 Applications of the Residue Formula

Example 19.2

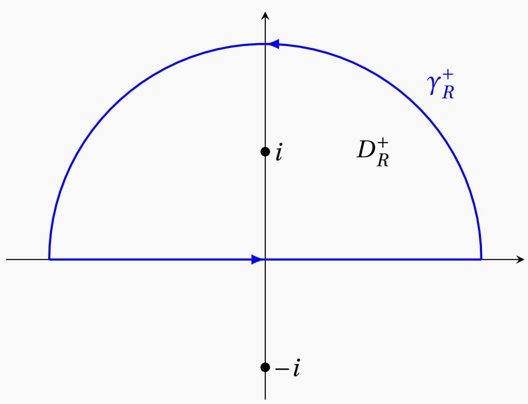
Consider the improper integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

From calculus, we know that

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x)|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Via complex numbers, we can consider the following semicircle:



The function $f(z) = \frac{1}{1+z^2}$ has simple poles at $z = \pm i$. From the residue formula, we know that

$$\oint_{\partial D_R^+} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}_i(f).$$

Calculating the residue, we just need to evaluate $(z-i)f(z)$ at $z = i$, since it is a simple pole:

$$\operatorname{Res}_i(f) = \left((z-i)\cdot\frac{1}{(z-i)(z+i)}\right)\bigg|_{z=i} = \frac{1}{z+i}\bigg|_{z=i} = \frac{1}{2i}.$$

Plugging this in, we know that

$$\oint_{\partial D_R^+} f(z)\,\mathrm{d}z = 2\pi i \cdot \frac{1}{2i} = \pi.$$

Further, we know that we can split ∂D_R^+ into two parts; one part along the real axis, and another curve above the real axis:

$$\oint_{\partial D_R^+} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{\gamma_R^+} f(z)\,\mathrm{d}z.$$

Taking the limit as $R \to \infty$ makes the first term $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x$, which is our desired term.

Our claim is then that $\int_{\gamma_R^+} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$. If we look at the norm, we have

$$\left|\int_{\gamma_R^+} \frac{1}{1+z^2}\,\mathrm{d}z\right| \le \sup_{z\in\gamma_R^+}\left|\frac{1}{1+z^2}\right| \cdot \pi R$$

Looking at the supremum, we can bound $\left|1+z^2\right| \ge |z|^2 - 1 > 0$, meaning $\left|\frac{1}{1+z^2}\right| \le \frac{1}{R^2-1}$.

This gives us $\frac{\pi R}{R^2-1}$ as a bound on the norm. Taking the limit as $R \to \infty$, this goes to zero; this means our final expression is

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \pi.$$

Example 19.3

Consider the improper integral

$$\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^n}\,\mathrm{d}x,$$

for $n \ge 1$. Here, there is no calculus method to evaluate the integral, but the complex method earlier still works.

We can see that $f(z) = \frac{1}{(1+z^2)^n}$ has two poles at $\pm i$, both with order n.

With the same semicircle from the previous example, we can see that

$$\oint_{\partial D_R^+} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}_i(f),$$

since the only pole inside D_R^+ is $z = i$.

We can again split up the LHS integral into two parts:

$$\oint_{\partial D_R^+} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{\gamma_R^+} f(z)\,\mathrm{d}z.$$

The first term is our desired expression as $R \to \infty$, and we will show the second term tends to zero as $R \to \infty$. In a similar fashion to the previous example, we have

$$\begin{aligned}\left|\int_{\gamma_R^+} \frac{1}{(1+z^2)^n}\,\mathrm{d}z\right| &\le \sup_{z\in\gamma_R^+}\left|\frac{1}{(1+z^2)^n}\right| \cdot \pi R \\ &\le \frac{\pi R}{(R^2-1)^n}\end{aligned}$$

The last term here is gotten through a very similar bound, except now we're just raising it to the power of n. Further, this does indeed go to zero as $R \to \infty$, so we have

$$\int_{-\infty}^{\infty} \frac{1}{(1+z^2)^n}\,\mathrm{d}x = 2\pi i \operatorname{Res}_i(f).$$

Calculating the residue, we have

$$\begin{aligned}
\operatorname{Res}_i(f) &= \frac{1}{(n-1)!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left((z-i)^n\cdot\frac{1}{(1+z^2)^n}\right)\bigg|_{z=i}\\
&= \frac{1}{(n-1)!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left(\frac{1}{(z+i)^n}\right)\bigg|_{z=i}\\
&= \frac{1}{(n-1)!}\,\frac{(-1)^{n-1}\cdot n(n+1)\cdots(n+(n-1)-1)}{(i+i)^{n+(n-1)}}\\
&= \frac{1}{(n-1)!}\,\frac{(-1)^{n-1}\cdot n(n+1)\cdots(2n-2)}{(2i)^{2n-1}}
\end{aligned}$$

In the denominator, we have $2n-1$ factors of i, which we can group into $n-1$ factors of $i^2=-1$ and one extra factor of i:

$$\begin{aligned}
&= \frac{1}{(n-1)!}\,\frac{\cancel{(-1)^{n-1}}\cdot n(n+1)\cdots(2n-2)}{2^{2n-1}\cdot\cancel{(-1)^{n-1}}\,i}\\
&= \frac{1}{(n-1)!}\,\frac{n(n+1)\cdots(2n-2)}{2^{2n-1}\,i}\\
&= \frac{1}{2i}\,\frac{(2n-3)!!}{(2n-2)!!}
\end{aligned}$$

Here, recall $n!! = n(n-2)(n-4)\cdots$; this means the numerator of the fraction has only odd terms, and the denominator of the fraction has only even terms.

Putting this all together, we have

$$\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)^n}\,\mathrm{d}x = \pi\cdot\frac{(2n-3)!!}{(2n-2)!!}.$$

Example 19.4

Consider the improper integral

$$\int_{-\infty}^{+\infty}\frac{e^{\frac{x}{2}}}{1+e^x}\,\mathrm{d}x.$$

Here, we can let $f(z)=\frac{e^{\frac{z}{2}}}{1+e^z}$. The poles of f occur when $1+e^z=0$; letting $z=x+iy$, we have

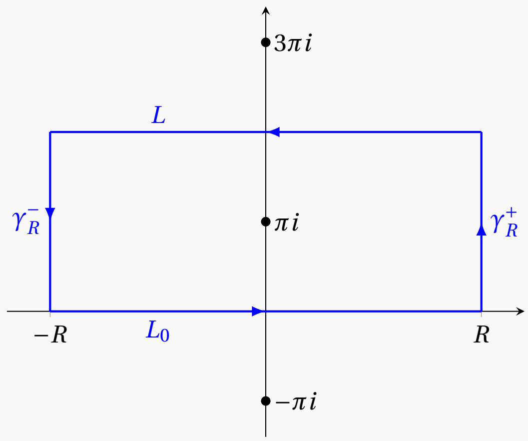
$$1+e^z=0 \implies e^z=-1 \implies e^x\cdot e^{iy}=1\cdot e^{i\pi}.$$

This means that

$$\begin{cases} e^x=1\\ y=\pi+2\pi k \quad k\in\mathbb{Z}\end{cases} \iff \begin{cases} x=0\\ y=\pi(2k+1) \quad k\in\mathbb{Z}\end{cases}.$$

As such, f has simple poles at $z=\pi(2k+1)i$ for $k\in\mathbb{Z}$.

Suppose we look at the following rectangle, Rec_R:



f has only one pole in Rec_R, namely $z = \pi i$; this means that

$$\int_{\partial \text{Rec}_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}_{\pi i}(f).$$

Splitting up the boundary of the rectangle, we have

$$\int_{\partial \text{Rec}_R} f(z)\,\mathrm{d}z = \int_{\gamma_R^+} f(z)\,\mathrm{d}z + \int_{\gamma_R^+} f(z)\,\mathrm{d}z + \int_L f(z)\,\mathrm{d}z + \int_{L_0} f(z)\,\mathrm{d}z.$$

We have

$$\begin{aligned}\left|\int_{\gamma_R^+} f(z)\,\mathrm{d}z\right| &\leq \sup_{z\in\gamma_R^+} |f(z)| \cdot 2\pi \\ &= \sup_{z\in\gamma_R^+} \left|\frac{e^{\frac{z}{2}}}{1+e^z}\right| \cdot 2\pi\end{aligned}$$

Here, we can parameterize γ_R^+ as $z = R + yi$ for $0 \leq y \leq 2\pi$, making $\left|e^{\frac{z}{2}}\right| = e^{\frac{R}{2}}$ and $|e^z| = e^R$. Further, the triangle inequality gives $|e^z + 1 - 1| \leq |e^z + 1| + 1 \implies |1 + e^z| \geq |e^z| - 1$. This means that we have

$$\leq \frac{e^{\frac{R}{2}}}{e^R - 1} \cdot 2\pi$$

This quantity goes to zero as $R \to \infty$.

A similar argument gives us

$$\lim_{R\to\infty} \left|\int_{\gamma_R^-} f(z)\,\mathrm{d}z\right| = 0.$$

Looking at the horizontal lines now, we can parameterize L as $z(t) = -t + 2\pi i$ for $t \in [-R, R]$, with $z'(t) = -1$.

This means that we have

$$\begin{aligned}\int_L f(z)\,\mathrm{d}z &= \int_{-R}^{R} \frac{e^{\frac{1}{2}(-t+2\pi i)}}{1+e^{-t+2\pi i}}\cdot(-1)\,\mathrm{d}t \\ &= \int_{-R}^{R} \frac{e^{-\frac{1}{2}t}e^{\pi i}}{1+e^{-t}e^{2\pi i}}\,\mathrm{d}t \\ &= \int_{-R}^{R} \frac{-e^{-\frac{t}{2}}}{1+e^{-t}}\cdot(-1)\,\mathrm{d}t \\ &= -\int_{R}^{-R} \frac{e^{\frac{x}{2}}}{1+e^{x}}\,\mathrm{d}x && (x=-t;\ \mathrm{d}x=-\mathrm{d}t) \\ &= \int_{-R}^{R} \frac{e^{\frac{x}{2}}}{1+e^{x}}\,\mathrm{d}x\end{aligned}$$

A similar argument gives the same result with

$$\int_{L_0} f(z)\,\mathrm{d}z = \int_{-R}^{R} \frac{e^{\frac{x}{2}}}{1+e^{x}}\,\mathrm{d}x.$$

This means that we have

$$\int_L f(z)\,\mathrm{d}z + \int_{L_0} f(z)\,\mathrm{d}z = 2\int_{-R}^{R} \frac{e^{\frac{x}{2}}}{1+e^{x}}\,\mathrm{d}x \to 2\int_{-\infty}^{\infty} \frac{e^{\frac{x}{2}}}{1+e^{x}}\,\mathrm{d}x,$$

where $R\to\infty$. Note that this is the integral that we originally want; we have

$$\int_{\partial\mathrm{Rect}_R} f(z)\,\mathrm{d}z = 2\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i\,\mathrm{Res}_{\pi i}(f).$$

Looking at the residue, since $z=\pi i$ is a simple pole, we have

$$\begin{aligned}\mathrm{Res}_{\pi i}(f) &= \big((z-\pi i)f(z)\big)\Big|_{z=\pi i} \\ &= \left((z-\pi i)\frac{e^{\frac{z}{2}}}{1+e^{z}}\right)\Bigg|_{z=\pi i} \\ &= \frac{(z-\pi i)e^{\frac{z}{2}}}{1+e^{(z-\pi i)+\pi i}}\Bigg|_{z=\pi i} \\ &= \frac{(z-\pi i)e^{\frac{z}{2}}}{1-e^{z-\pi i}}\Bigg|_{z=\pi i} \\ &= \frac{(z-\pi i)e^{\frac{z}{2}}}{\cancel{1}-\left(\cancel{1}+(z-\pi i)+\frac{(z-\pi i)^2}{2!}+\frac{(z-\pi i)^3}{3!}+\cdots\right)}\Bigg|_{z=\pi i} \\ &= \frac{\cancel{(z-\pi i)}e^{\frac{z}{2}}}{-\cancel{(z-\pi i)}\left(1+\frac{(z-\pi i)}{2!}+\frac{(z-\pi i)^2}{3!}+\cdots\right)}\Bigg|_{z=\pi i} \\ &= \frac{e^{\frac{\pi i}{2}}}{-1} = -e^{\frac{\pi i}{2}} = -i\end{aligned}$$

Plugging this in, we have

$$\int_{-\infty}^{\infty} \frac{e^{\frac{x}{2}}}{1+e^{x}}\,\mathrm{d}x = \pi i\,\mathrm{Res}_{\pi i}(f) = \pi i(-i) = \boxed{\pi}.$$

An extension of this example is to consider the more general form

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,\mathrm{d}x,$$

for $0<a<1$. It turns out that this integral evaluates to $\frac{\pi}{\sin(a\pi)}$, but is left as an exercise.

4/5/2022

Lecture 20

Singularities and Meromorphic Functions

Theorem 20.1: Riemann's Theorem on Removable Singularities

Suppose we have an open set $\Omega \subseteq \mathbb{C}$, with $z_0 \in \Omega$. Further, suppose $f(z)$ is a holomorphic function on $\Omega \setminus \{z_0\}$.

If f is bounded near z_0, then z_0 is a removable singularity, i.e. f can be holomorphically extended to Ω.

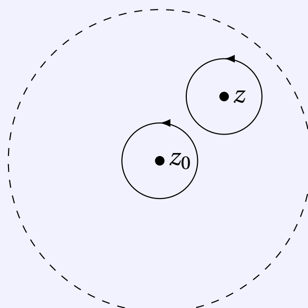
Proof. Suppose we take a small disk $\overline{D(z_0)} \subseteq \Omega$; we can try to extend f holomorphically to the whole disk. That is, we want a $g(z)$ that is holomorphic on $D(z_0)$, where $g(z) = f(z)$ on $D^*(z_0) = D(z_0) \setminus \{z_0\}$.

How do we define $g(z)$? Since g is holomorphic on $D(z_0)$, we can use Cauchy's integral formula to say

$$g(z) = \frac{1}{2\pi i} \int_{\partial D(z_0)} \frac{g(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i} \int_{\partial D(z_0)} \frac{f(w)}{w-z}\,\mathrm{d}w,$$

where the last equality is due to the fact that $g(z) = f(z)$ on the boundary.

Now, we just need to show that $g(z) = f(z)$ for any $z \in D^*(z_0)$. Let us consider two small disks $D_\varepsilon(z_0)$ and $D_r(z)$ (here, $z \in D^*(z_0)$ is arbitrary), with radii $r > 0$ and $\varepsilon > 0$ such that the small disks do not intersect.



Since $\frac{f(w)}{w-z}$ is holomorphic on $D(z_0) \setminus (D_r(z) \cup D_\varepsilon(z_0))$, we can use Cauchy's theorem to say

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_{\partial D(z_0)} \frac{f(w)}{w-z}\,\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D_\varepsilon(z_0)} \frac{f(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i} \int_{\partial D_r(z)} \frac{f(w)}{w-z}\,\mathrm{d}w \end{aligned}$$

The second term is equal to $f(z)$ by Cauchy's integral formula, since f is holomorphic around $z \in D^*(z_0)$.

As such, we just need to show that

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\partial D_\varepsilon(z_0)} \frac{f(w)}{w-z}\,\mathrm{d}w = 0.$$

Looking at the norm, we have

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\partial D_\varepsilon(z_0)} \frac{f(w)}{w-z}\,\mathrm{d}w \right| &\leq \frac{1}{2\pi} \sup_{w \in \partial D_\varepsilon(z_0)} \frac{|f(w)|}{|w-z|} \cdot 2\pi\varepsilon \\ &= \varepsilon \sup_{W \in \partial D_\varepsilon(z_0)} \\ &\leq \varepsilon \frac{C}{R} \end{aligned}$$

Here, the last inequality is due to the fact that f is bounded near z_0 (i.e. $|f(w)| \le C$), and the fact that $|w-z|$ is bounded by the radius of the outer circle (i.e. $|w-z| \le R$). This last quantity goes to 0 as $\varepsilon \to 0^+$.

As such, we've just shown that $g(z) = f(z)$ on $D^*(z_0)$, and as such z_0 is a removable singularity. □

Corollary 20.2

Suppose we have an open set $\Omega \subseteq \mathbb{C}$, with $z_0 \in \Omega$. Further suppose f is holomorphic on $\Omega \setminus \{z_0\}$.

z_0 is a pole if and only if

$$\lim_{z\to z_0} |f(z)| = +\infty.$$

Proof. ($\Longrightarrow$) Suppose z_0 is a pole. We'll show that

$$\lim_{z\to z_0} |f(z)| = +\infty.$$

We know that for a pole z_0, we can write f near z_0 as

$$f(z) = \frac{g(z)}{(z-z_0)^n},$$

where n is the order of the pole z_0, and $g(z)$ is holomorphic and nowhere vanishing near z_0.

Looking at the limit, we have

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \frac{|g(z)|}{|z-z_0|^n} = +\infty,$$

since the numerator is continuous at z_0 and specifically $|g(z_0)| \neq 0$, and the denominator approaches 0.

($\Longleftarrow$) Suppose that $\lim_{z\to z_0} |f(z)| = +\infty$. We'll show that z_0 is a pole singularity.

Firstly, $\lim_{z\to z_0} |f(z)| = +\infty$ implies that $f(z) \neq 0$ near z_0, so we can consider $g(z) = \frac{1}{f(z)}$ near z_0, excluding z_0.

We know that g is holomorphic on the punctured disk centered at z_0, so z_0 is an isolated singularity of g. Further, we have that

$$|g(z)| = \frac{1}{|f(z)|},$$

so g is bounded near z_0, since $\lim_{z\to z_0} |f(z)| = +\infty$.

By Riemann's theorem on removable singularities, g has z_0 as a removable singularity, so it can be extended holomorphically to z_0; let $\hat{g}(z)$ denote this extension.

Looking at $\hat{g}$, we have

$$\hat{g}(z_0) = \lim_{z\to z_0} \hat{g}(z) = \lim_{z\to z_0} g(z) = \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

This means that $\hat{g}(z)$ has z_0 as a zero, and by definition $f(z)$ must have z_0 as a pole. □

Corollary 20.3

Further, an isolated singularity z_0 can be classified as

- *removable*, if f is bounded near z_0
- *a pole*, if $\lim_{z\to z_0} |f(z)| = +\infty$

- *essential*, otherwise.

These last few theorems/corollaries show us that the behavior of $|f(z)|$ near z_0 tells us the type of singularity of z_0.

20.1 Essential Singularities

Suppose we consider $f(z) = e^{\frac{1}{z}}$, which is holomorphic on $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$.

It turns out that $z_0 = 0$ is an essential singularity. It is not a removable singularity, since f is not bounded near zero; it also isn't a pole, since $|f(z)|$ does not diverge to $+\infty$ (the limit doesn't exist). This means that $z_0 = 0$ is an essential singularity.

Theorem 20.4: Great Picard's Theorem

Suppose f is a holomorphic function on $D^*(z_0) = D(z_0) \setminus \{z_0\}$, with z_0 as an essential singularity.

Then, f takes every point in $\mathbb{C}$ infinitely many times with at most one exception.

An example of Great Picard's theorem can be seen with $f(z) = e^{\frac{1}{z}}$. f cannot take on the value of 0, but for any other $w \neq 0$, the function $e^{\frac{1}{z}} = w$ infinitely many times (i.e. it has infinitely many solutions).

This theorem is not easy to prove, but we will state a weaker version that is easier to prove.

Theorem 20.5: Casorati–Weierstrass Theorem

Suppose f is holomorphic on $D^*(z_0)$ with z_0 as an essential singularity.

Then $f(D^*(z_0))$ is dense in $\mathbb{C}$.

Proof. Suppose for contradiction that $f(D^*(z_0))$ is *not* dense in $\mathbb{C}$, and we will argue that z_0 cannot be an essential singularity.

Since $f(D^*(z_0))$ is not dense, we can find some $w_0 \in \mathbb{C}$ and $\varepsilon > 0$ such that $D_\varepsilon(w_0) \cap f(D^*(z_0)) = \varnothing$.

Let us consider $g(z) = \frac{1}{f(z) - w_0}$; this function is defined for any $z \in D^*(z_0)$, and is holomorphic on $D^*(z_0)$.

This means that we can consider z_0 as an isolated singularity of $g(z)$.

We claim that $g(z)$ is bounded on $D^*(z_0)$:

$$|f(z) - w_0| \geq \varepsilon \implies |g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\varepsilon}.$$

By Riemann's theorem on removable singularities, we can determine that z_0 is a removable singularity of $g(z)$, with $\hat{g}(z)$ as its extension.

There are now two cases. If $\hat{g}(z_0) \neq 0$, then $f(z)$ has a removable singularity at z_0. If $\hat{g}(z_0) = 0$, then $f(z)$ has a pole at z_0. In any case, z_0 must not be an essential singularity of f—this is a contradiction.

As such, it must have been the case that $f(D^*(z_0))$ is indeed dense in $\mathbb{C}$. □

4/7/2022

Lecture 21

Meromorphic Functions

Definition 21.1: Meromorphic Function

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with f defined on $\Omega \setminus \{z_1, z_2, \dots\}$, where $\{z_1, z_2, \dots\}$ is a sequence of distinct points in Ω, with no limit point in Ω.

Further, if f is holomorphic on $\Omega \setminus \{z_1, z_2, \dots\}$ with $\{z_1, z_2, \dots\}$ as pole singularities, then f is a *meromorphic function* on Ω.

One special case is if $\{z_1, z_2, \dots\} = \varnothing$; in this case, f is both meromorphic and holomorphic.

Another special case is if $\{z_1, z_2, \dots, z_n\}$ is a finite sequence; if f has $\{z_1, \dots, z_n\}$ as poles, then f is meromorphic.

Example 21.2

Suppose $\Omega = \mathbb{C}$, and we consider

$$f(z) = \frac{1}{z(z-1)}.$$

Here, f is not holomorphic on $\mathbb{C}$, but it is meromorphic on $\mathbb{C}$, since it has $z_1 = 0$ and $z_2 = 1$ as pole singularities.

As a generalization, suppose we extend the complex plane $\mathbb{C}$ to the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Consider some meromorphic function $f : \Omega \to \overline{\mathbb{C}}$; we can define it with

$$\begin{cases} f(z) \in \mathbb{C} & z \in \Omega \setminus \{z_1, z_2, \dots\} \\ f(z) = \infty \in \overline{\mathbb{C}} & z \in \{z_1, z_2, \dots\} \end{cases}.$$

Here, $\infty \notin \mathbb{C}$; it is a part of the extended complex plane. In this sense, we can look at meromorphic functions as having a codomain of the extended complex plane.

In a similar manner, we can also extend the domain to the extended complex plane, though we'll first look at the topology of the extended complex plane.

In topology, we are concerned with defining the neighborhoods of points; in the complex plane, the neighborhood of a point z_0 can just be the disk $D_\varepsilon(z_0)$. With the extended complex plane, we also have the case $z_0 = \infty$, which we need to take care of as well.

Here, we define $D_R(\infty)$ for $R > 0$ as the set $\{z \in \mathbb{C} \mid |z| > R\} \cup \{\infty\}$, as shown in Fig. 21.1

Definition 21.3: Open set

A subset Ω of $\overline{\mathbb{C}}$ is an *open set* if for every point $z_0 \in \Omega$, we can find some $R > 0$ such that $D_R(z_0) \subseteq \Omega$.

This is exactly the same as the definition of an open set in $\mathbb{C}$; we've just extended this definition to also include $z_0 = \infty$.

Example 21.4

Suppose we have $\Omega = \{z \mid |z| > 1\} \cup \{\infty\} \subseteq \overline{\mathbb{C}}$.

We claim that Ω is an open set in $\overline{\mathbb{C}}$.

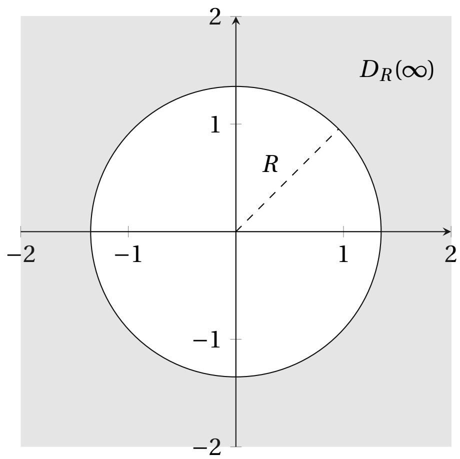


Figure 21.1: Disk at ∞ in the extended complex plane.

To see this, suppose we take any $z_0 \in \Omega$. If $z_0 \neq \infty$, then $D_\varepsilon \subseteq \{z \mid |z| > 1\}$ for small enough $\varepsilon > 0$. If $z_0 = \infty$, then $D_R(\infty) \subseteq \Omega$ for any $R > 1$.

With this extension, we can talk about meromorphic functions on the extended complex plane $\overline{\mathbb{C}}$, with the exact same definition:

Definition 21.5: Meromorphic Function (on extended complex plane)

Suppose $\Omega \subseteq \overline{\mathbb{C}}$ is an open set, with f defined on $\Omega \setminus \{z_1, z_2, \ldots\}$, where $\{z_1, z_2, \ldots\}$ is a sequence of distinct points in Ω, with no limit point in Ω.

Further, if f is holomorphic on $\Omega \setminus \{z_1, z_2, \ldots\}$ with $\{z_1, z_2, \ldots\}$ as pole singularities, then f is a *meromorphic function* on Ω.

Definition 21.6: Singularities at ∞

Suppose we have a function f defined on $D_r(\infty) \setminus \{\infty\}$ for some $R > 0$, where f is also holomorphic on its domain.

∞ is always a singularity, but we can classify the singularity as well, through the usage of $\hat{f}(z) = f(\frac{1}{z})$:

- *removable singularity*, if $\hat{f}(z)$ has 0 as a removable singularity
- *pole singularity*, if $\hat{f}$ has 0 as a pole singularity
- *essential singularity*, if $\hat{f}$ has 0 as an essential singularity.

Example 21.7

Suppose we consider $f(z) = z$. We know that f is holomorphic on $\mathbb{C}$; let us consider f on $\overline{\mathbb{C}}$.

f has ∞ as a singularity; it has no other singularity approaching ∞, so it is an isolated singularity.

Let us consider $\hat{f}(z) = f(\frac{1}{z}) = \frac{1}{z}$; it has 0 as a simple pole, so ∞ is a simple pole of $f(z) = z$.

Example 21.8

Consider the function $f(z) = \frac{1}{z}$; discuss the singularities of f on $\overline{\mathbb{C}}$.

At $z_1 = 0$, f has an isolated singularity, and $z_1 = 0$ a simple pole of f.

At $z_2 = \infty$, f also has an isolated singularity, and $\hat{f}(z) = z$; 0 is a removable singularity of $\hat{f}$, so ∞ is a removable singularity of f.

Example 21.9

Consider the function

$$f(z) = \frac{1}{z^3} + \frac{1}{z-i} + z^2 + z + 1.$$

Discuss the singularities of f in $\overline{\mathbb{C}}$.

We can see that f has singularities at $z_1 = 0$, $z_2 = i$, and $z_3 = \infty$.

Further, we know that $z_1 = 0$ is an order 3 pole, and $z_2 = i$ is a simple pole.

Looking at $z_3 = \infty$, we have

$$\begin{aligned}\hat{f}(z) &= \frac{1}{\left(\frac{1}{z}\right)^3} + \frac{1}{\frac{1}{z} - i} + \left(\frac{1}{z}\right)^2 + \frac{1}{z} + 1 \\ &= z^3 + \frac{iz}{z+i} + \frac{1}{z^2} + \frac{1}{z} + 1\end{aligned}$$

We can see that 0 is an order 2 pole of $\hat{f}$ (because of the $\frac{1}{z^2}$ term), so $z_3 = \infty$ is an order 2 pole of f.

Further, since all of these singularities are poles, $f(z)$ is meromorphic on $\overline{\mathbb{C}}$.

Example 21.10

Consider the function $f(z) = e^z$.

The only singularity of f on $\overline{\mathbb{C}}$ is ∞; let us see what type of singularity it is.

We have $\hat{f}(z) = e^{\frac{1}{z}}$; $\hat{f}$ has 0 as an essential singularity (the limit as $z \to 0$ doesn't exist).

Here, this is an example where f is entire (i.e. holomorphic on $\mathbb{C}$), but not a meromorphic function on $\overline{\mathbb{C}}$.

Theorem 21.11: Meromorphic and rational functions

A function f is meromorphic on $\overline{\mathbb{C}}$ if and only if it is a rational function.

Proof. ($\Longleftarrow$) We want to show that every rational function is meromorphic on $\overline{\mathbb{C}}$.

Here, we can write any rational function as

$$f(z) = \frac{p(z)}{q(z)},$$

where p and q are polynomial functions, assuming p and q have no common factors.

f has singularities at $\{z \mid q(z) = 0\}$ (which is a finite set), and it also has a singularity at ∞.

Looking at $\hat{f}$, we have

$$\hat{f}(z) = \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)},$$

which is also a rational function. This means that 0 is either a removable singularity or a pole singularity (rational functions do not have essential singularities in $\mathbb{C}$). This means that ∞ is either removable or a pole singularity of $f(z)$ (this can be found by determining whether the norm of $\hat{f}$ is bounded or not; if it is bounded, then 0 is removable, and if not, then 0 is a pole).

In any case, the singularities are either removable or poles, and as such f is meromorphic.

($\implies$) Suppose that f is meromorphic on $\overline{\mathbb{C}}$; we will show that f must be a rational function.

By the definition of a meromorphic function, we know that f has pole singularities $z_1, z_2, \ldots \in \mathbb{C}$; we claim that there must be finitely many pole singularities.

To see this, suppose for contradiction that there were infinitely many such pole singularities, and look at the sequence $\{z_n\}$.

If this sequence is bounded, then there exists some convergent subsequence, which contradicts the definition of a meromorphic function (it'd have a limit point). If this sequence is unbounded, then there is some convergent subsequence in $\overline{\mathbb{C}}$ to ∞, which also contradicts the definition of a meromorphic function on $\overline{\mathbb{C}}$.

In either case, having infinitely many pole singularities would contradict the definition of a meromorphic function on $\overline{\mathbb{C}}$, and as such there must only be finitely many pole singularities.

Now, suppose we look at the neighborhood of the pole singularities z_j for $j = 1, \ldots, n$ (here, we assume that there are n pole singularities).

At each one of these singularities (by Theorem 18.10), we can write f in terms of its principal part and its holomorphic part:

$$f(z) = f_j(z) + g_j(z).$$

Here, we let f_j be the principal function and let g_j be the holomorphic function in the decomposition at the pole singularity.

Similarly, near ∞, we can use the same decomposition with $\hat{f}$:

$$\hat{f}(z) = f\left(\frac{1}{z}\right) = \hat{f}_\infty(z) + \hat{g}_\infty(z).$$

Here, $\hat{f}$ has a pole (or removable) singularity at 0, so we can decompose $\hat{f}$ at $z = 0$. (One note here is that $\hat{f}_\infty(z)$ may be zero in the case that $\hat{f}$ has a removable singularity at 0; in contrast, f will always have a principal part at all other pole singularities in $\mathbb{C}$.)

Here, we further define $f_\infty(z) = \hat{f}_\infty\left(\frac{1}{z}\right)$ and $g_\infty(z) = \tilde{g}_\infty\left(\frac{1}{z}\right)$.

Suppose we consider the function

$$h(z) = f(z) - \big(f_1(z) + f_2(z) + \cdots + f_n(z)\big) - f_\infty(z).$$

We know that h is holomorphic on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$; our claim is that $z_1, z_2, \ldots, z_n$ are all removable singularities of $h(z)$.

To see this, consider some $j_0 \in \{1, 2, \ldots, n\}$, and look at $h(z)$ near z_{j_0}:

$$\begin{aligned} h(z) &= f(z) - f_{j_0}(z) - \big(f_1(z_0) + \cdots f_{j_0-1}(z) + f_{j_0+1}(z) + \cdots + f_n(z)\big) - f_\infty(z) \\ &= g_{j_0}(z) - \big(f_1(z_0) + \cdots + f_{j_0-1}(z) + f_{j_0+1}(z) + \cdots + f_n(z)\big) - f_\infty(z) \end{aligned}$$

Here, we can notice a few things:

- $g_{j_0}(z)$ is holomorphic in a neighborhood of z_{j_0} by definition (it's the holomorphic part of f).
- $f_1(z) + \cdots + f_{j_0-1}(z) + f_{j_0+1}(z) + \cdots + f_n(z)$ can each be expressed as a sum of terms in the form $\frac{?}{(z-z_j)^?}$, for $z_j \neq z_{j_0}$.

  This means that each of these terms are also holomorphic in a neighborhood of z_{j_0}, since their singularities are far away from z_{j_0}.
- $f_\infty(z)$ is also holomorphic in a neighborhood of z_{j_0}.

Hence, $h(z)$ is bounded in a neighborhood of z_{j_0}, since it is holomorphic in a neighborhood of z_{j_0}. By Riemann's theorem on removable singularities, $h(z)$ has a removable singularity at z_{j_0}.

This applies to any $z_{j_0} \in \{z_1, \ldots, z_n\}$, so each one of these singularities are removable singularities of h.

This means that after a holomorphic extension, h is an entire function.

We further claim that $h(z)$ is bounded on $\mathbb{C}$. To see this, we can use a similar method as before:

$$\begin{aligned} h(z) &= f(z) - f_\infty(z) - \big(f_1(z) + f_2(z) + \cdots + f_n(z)\big) \\ &= g_\infty(z) - \big(f_1(z) + f_2(z) + \cdots + f_n(z)\big) \end{aligned}$$

We know that $g_\infty(z)$ is bounded at ∞, and each $f_j(z)$ term is also bounded at ∞ since they are fractions with z in the denominator much like earlier.

This means that there exists an $R > 0$ such that $\left|g_\infty(z)\right|$ is bounded on $\mathbb{C} \setminus D_r(0)$, so $h(z)$ is bounded.

By Liouville's theorem, we can then conclude that $h(z)$ is a constant function. Looking at our prior definition of $h(z)$, which we now know to be some constant $c \in \mathbb{C}$, we have

$$\begin{aligned} c = h(z) &= f(z) - \big(f_1(z) + f_2(z) + \cdots f_n(z)\big) - f_\infty(z) \\ f(z) &= f_1(z) + f_2(z) + \cdots + f_n(z) + f_\infty(z) + c \\ &= \left(\frac{c_{-k_1}}{(z-z_1)^{k_1}} + \frac{c_{-(k_1-1)}}{(z-z_1)^{k_1-1}} + \cdots + \frac{c_{-1}}{z-z_1}\right) + \left(c_{k_\infty} z^{k_\infty} + c_{k_\infty-1} z^{k_\infty-1} + \cdots + c_1 z\right) \end{aligned}$$

This is in the form of a rational function, so we've shown that f is indeed a rational function. □

4/12/2022

Lecture 22

Argument Principle

Firstly, we'll describe a geometric interpretation of the extended complex plane $\overline{\mathbb{C}}$.

The extended complex plane comes naturally through a stereographic projection, as shown in Fig. 22.1.

Here, we have a sphere of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$. Starting at the north pole $(0, 0, 1)$, we can draw a line to some point (x, y) in the complex plane. This line intersects the sphere at some point (X, Y, Z), and we call (x, y) the *stereographic projection* of (X, Y, Z).

Formally, we can let $\mathcal{S}$ denote the sphere, i.e. $\partial B_{\frac{1}{2}}(0, 0, \frac{1}{2})$. The stereographic projection is a bijective map π_N relative to the north pole $N = (0, 0, 1)$:

$$\pi_N : \mathcal{S} \setminus \{N\} \to \mathbb{C}.$$

We can extend this map to include the north pole:

$$\hat{\pi}_N : \mathcal{S} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Specifically, this stereographic projection π_N maps N to ∞ in the extended complex plane.

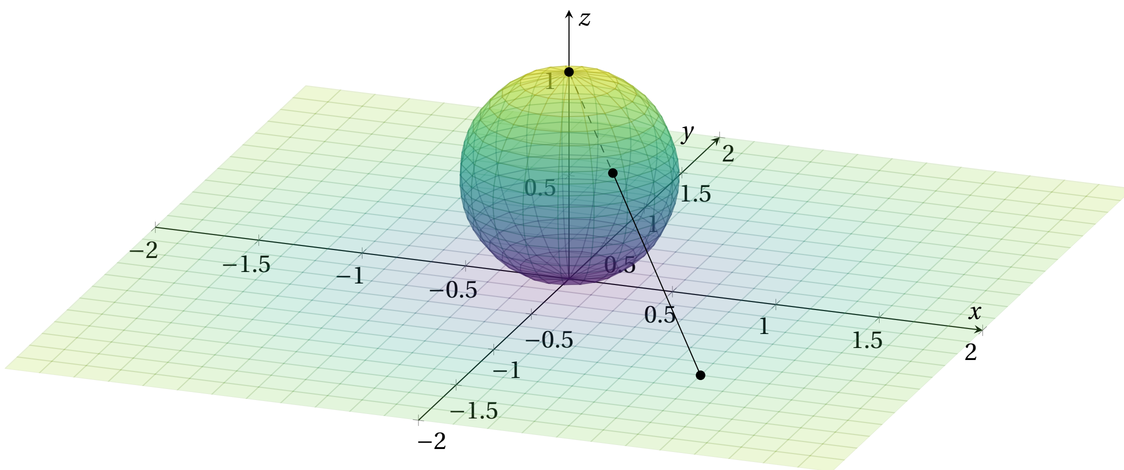


Figure 22.1: Stereographic projection of a sphere onto a plane.

We'll talk about this later, but the stereographic projection is a *conformal mapping* as well.

Extending this idea, we can also look at a stereographic projection from the south pole $S = (0,0,0)$ as well. This projection would be on the plane $z = 1$; the mappings are shown below:

$$\pi_N : \mathcal{S} \to \overline{\mathbb{C}} \qquad \pi_S : \mathcal{S} \to \mathbb{C}$$

$$\pi_N : (X,Y,Z) \mapsto \underbrace{\frac{X}{Z}}_{x} \underbrace{-\frac{Y}{Z}}_{y} i \qquad \pi_S : (X,Y,Z) \mapsto \underbrace{\frac{X}{1-Z}}_{x'} + \underbrace{\frac{Y}{1-Z}}_{y'} i$$

Here, we denote the output of π_N as w, and the output of π_S as w'. It turns out that $ww' = 1$:

$$\begin{aligned} ww' &= \left(\frac{X}{Z} - \frac{Y}{Z} i\right)\left(\frac{X}{1-Z} + \frac{Y}{1-Z} i\right) \\ &= \frac{X^2}{(1-Z)Z} + \frac{Y^2}{(1-Z)z} + \left(\frac{YX}{(1-Z)Z} - \frac{XY}{(1-Z)Z}\right) i \\ &= \frac{X^2 + Y^2}{(1-Z)Z} = 1 \end{aligned}$$

Since (X,Y,Z) lies on a sphere, we must have $X^2 + Y^2 + (Z - \frac{1}{2})^2 = \left(\frac{1}{2}\right)^2$; this simplifies to give $X^2 + Y^2 = Z(1-Z)$, which means that ww' does indeed equal 1.

This explains why the neighborhood of ∞ in $f(z)$ corresponds to the neighborhood of 0 in $\hat{f}(z) = f\left(\frac{1}{z}\right)$.

22.1 Argument Principle

We'll motivate the argument principle through a few examples.

Example 22.1

Consider the function $f(z) = z^n$; this function is holomorphic on $\mathbb{C}$, and has $z_0 = 0$ as the only zero (of order n).

We know the derivative is $f'(z) = nz^{n-1}$; dividing by $f(z)$, we have

$$\frac{f'(z)}{f(z)} = \frac{nz^{n-1}}{z^n} = \frac{n}{z}.$$

Taking the contour integral, we have

$$\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \oint_\gamma \frac{n}{z}\,\mathrm{d}z = n\cdot 2\pi i.$$

This means that we can recover the order of the zero by integration:

$$n = \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Example 22.2

Consider the function $f(z) = \frac{1}{z^n}$; this function is holomorphic on $\mathbb{C}^*$, and has $z_0 = 0$ as the only pole (of order n).

We know the derivative is $f'(z) = -nz^{-n-1}$; dividing by $f(z)$, we have

$$\frac{f'(z)}{f(z)} = \frac{-nz^{-n-1}}{z^{-n}} = -\frac{n}{z}.$$

Taking the contour integral, we have

$$\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \oint_\gamma -\frac{n}{z}\,\mathrm{d}z = -n\cdot 2\pi i.$$

This means that we can recover the order of the pole by integration as well:

$$-n = \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

These two examples give us motivation for the *argument principle*.

Theorem 22.3: Argument Principle

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with D as a disk such that $\overline{D} \subseteq \Omega$. Further, suppose f is a meromorphic function on Ω and has neither zeros nor poles on ∂D. Then,

$$\frac{1}{2\pi i}\oint_{\partial D} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \mathbf{n}_0 - \mathbf{n}_p,$$

where $\mathbf{n}_0$ is the sum of the orders of zeros of f inside D, and $\mathbf{n}_p$ is the sum of the orders of poles of f inside D.

Proof. Suppose we taken $\varepsilon > 0$ small enough such that $D_\varepsilon(z_n)$ for each zero or pole z_n is disjoint from each other, and all of the disks are inside D.

We know that $\frac{f'(z)}{f(z)}$ is holomorphic on $D' \setminus \bigcup_n \overline{D}_\varepsilon(z_n)$, where all z_n are poles or zeroes of f.

By Cauchy's theorem, we can see that

$$\oint_{\partial D} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_n \oint_{\partial D_\varepsilon(z_n)} \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Near every zero z_0, when $\varepsilon > 0$ is small enough, we can express $f(z) = (z - z_0)^{n_0} g(z)$ on $D_\varepsilon(z_0)$, where $g(z)$ is nowhere vanishing on $\overline{D_\varepsilon(z_0)}$. Here, n_0 is the order of the zero z_0.

Evaluating $\frac{f'(z)}{f(z)}$ on $D_\varepsilon(z_0)^*$, we have

$$\begin{aligned} f'(z) &= n_0(z-z_0)^{n_0-1}g(z) + (z-z_0)^{n_0}g'(z) \\ \frac{f'(z)}{f(z)} &= \frac{n_0(z-z_0)^{n_0-1}g(z) + (z-z_0)^{n_0}g'(z)}{(z-z_0)^{n_0}g(z)} \\ &= \frac{n_0}{z-z_0} + \frac{g'(z)}{g(z)} \end{aligned}$$

We can see that the last term $\frac{g'(z)}{g(z)}$ is holomorphic on $\overline{D_\varepsilon(z_0)}$, since g is nowhere vanishing on the same domain.

This means that

$$\begin{aligned} \int_{\partial D_\varepsilon(z_0)} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \int_{\partial D_\varepsilon(z_0)} \frac{n_0}{z-z_0}\,\mathrm{d}z + \cancel{\int_{\partial D_\varepsilon(z_0)} \frac{g'(z)}{g(z)}\,\mathrm{d}z} \\ &= n_0 \cdot 2\pi i \end{aligned}$$

The last integral in the first line goes to zero, since the integrand is holomorphic.

If we sum all contributions from the zeros of f, we have

$$\begin{aligned} \sum_{z_0} \int_{\partial D_\varepsilon(z_0)} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= 2\pi i \cdot \sum_{z_0} n_0 \\ &= 2\pi i \cdot \mathbf{n}_0 \end{aligned}$$

Similarly, near every pole z_p, when $\varepsilon > 0$ is small enough, we can express $f(z) = \frac{g(z)}{(z-z_p)^{n_p}}$ on $D_\varepsilon(z_p)$, where $g(z)$ is nowhere vanishing on $\overline{D_\varepsilon(z_p)}$. Here, n_p is the order of the pole z_p.

Evaluating $\frac{f'(z)}{f(z)}$ on $D_\varepsilon(z_p)^*$, we have

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{-n_p(z-z_p)^{-n_p-1}g(z) + (z-z_p)^{-n_p}g'(z)}{(z-z_p)^{-n_p}g(z)} \\ &= \frac{-n_p}{z-z_p} + \frac{g'(z)}{g(z)} \end{aligned}$$

Again, $\frac{g'(z)}{g(z)}$ is holomorphic on $\overline{D_\varepsilon(z_p)}$ since g is nowhere vanishing on the same domain.

This means that

$$\begin{aligned} \int_{\partial D_\varepsilon(z_p)} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \int_{\partial D_\varepsilon(z_p)} \frac{-n_p}{z-z_p}\,\mathrm{d}z + \cancel{\int_{\partial D_\varepsilon(z_p)} \frac{g'(z)}{g(z)}\,\mathrm{d}z} \\ &= -2\pi i \cdot n_p \end{aligned}$$

If we sum all contributions from the poles of f, we have

$$\begin{aligned} \sum_{z_p} \int_{\partial D_\varepsilon(z_p)} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= -2\pi i \cdot \sum_{z_p} n_p \\ &= -2\pi i \cdot \mathbf{n}_p \end{aligned}$$

Adding all equalities, we have

$$\begin{aligned} \int_{\partial D} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= 2\pi i \mathbf{n}_0 - 2\pi i \mathbf{n}_p \\ \frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \mathbf{n}_0 - \mathbf{n}_p \end{aligned}$$

□

4/14/2022

Lecture 23

Argument Principle, Rouche's Theorem

Theorem 23.1: Rouche's Theorem

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with $\overline{D} \subseteq \Omega$. Consider two holomorphic functions f and g on Ω.

If $\left|f(z)\right| > \left|g(z)\right|$ on ∂D, then $f(z)$ and $f(z)+g(z)$ have an equal number of zeros (counting with orders).

Proof. Consider a *homotopy*

$$f_t(z) = f(z) + tg(z),$$

for $t \in [0,1]$.

This is a $[0,1]$-family of holomorphic functions on Ω, with

$$\begin{aligned} f_0(z) &= f(z) + 0 \cdot g(z) = f(z) \\ f_1(z) &= f(z) + 1 \cdot g(z) = f(z) + g(z) \end{aligned}$$

We'll now apply the argument principle to $f_t(z)$ on $\overline{D}$.

Our claim is that $f_t(z)$ has no zeroes on ∂D.

We can prove this claim through contradiction; suppose $f_t(z) = 0$ at some point on ∂D.

This means that

$$f_t(z) = f(z) + tg(z) = 0 \implies f(z) = -tg(z) \implies \left|f(z)\right| = t \cdot \left|g(z)\right|.$$

However, we know that $\left|f(z)\right| > \left|g(z)\right|$ on ∂D, which means that we must have $t \cdot \left|g(z)\right| > g(z)$.

This is not possible; if $g(z) = 0$, then both sides are zero and the inequality is not satisfied; if $g(z) \neq 0$, then we must have $t > 1$, which is not possible, contradicting with $t \in [0,1]$. This proves the claim by contradiction.

Using this claim, we can use the argument principle on $f_t(z)$. One remark to make here is that since f_t is holomorphic, it has no poles; we only need to consider the zeros of f_t.

The argument principle states that

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f_t'(z)}{f_t(z)} \, \mathrm{d}z = n_t,$$

where n_t is the number of zeros of f_t inside D.

Note that the LHS integral continuously depends on $t \in [0,1]$, but the RHS must be an integer, as it counts the number of zeros of f_t. This means that it must be the case that $n_0 = n_1$.

More rigorously, if $n_0 \neq n_1$, then IVT states that there must be some $t_0 \in [0,1]$ such that $n_{t_0} = m$ for $n_0 < m < n_1$ and $m \notin \mathbb{Z}$. This can't be possible though, since n_{t_0} must be an integer by the argument principle □

Example 23.2

Consider the function

$$p(z) = z^5 + z^3 - iz + 3i.$$

How many solutions are there to $p(z) = 0$ on $\mathbb{C}$?

Notice that the last part of the polynomial is essentially just a small perturbation of z^5.

Consider a big disk $D_R(0)$. We know that

$$\left|z^3 - iz + 3i\right| < \left|z^5\right|$$

on $\partial D_R(0)$. This means that Rouche's theorem says that the functions z^5 and $z^5 + (z^3 - iz + 3i) = p(z)$ have an equal number of zeros.

Since $z = 0$ is an order 5 zero of z^5, $p(z) = 0$ also has 5 roots.

Example 23.3

We give an alternate proof of the fundamental theorem of algebra with Rouche's theorem.

That is, we will show that $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + z_1 z + a_0$ for $a_n \neq 0$ and $n \geq 1$ has n roots on $\mathbb{C}$.

Here, suppose we consider $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

When $R > 0$ is large enough, then on $\partial D_R(0)$ (i.e. $|z| = R$), we have

$$\begin{aligned} \left|a_n z^n\right| &> \left|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right| \\ \left|f(z)\right| &> \left|g(z)\right| \end{aligned}$$

By Rouche's theorem, $p(z) = f(z) + g(z)$ has an equal number of zeros as $f(z)$.

Since $f(z) = a_n z^n = 0$ has n zeros, it must then be the case that $p(z) = 0$ also has n zeros.

4/19/2022

Lecture 24

Open Mapping Theorem, Maximum Modulus Principle, Complex Logarithm

Theorem 24.1: Open Mapping Theorem

Suppose $\Omega \subseteq \mathbb{C}$ is an open set.

If f is a nonconstant holomorphic function on Ω, then f is an *open mapping*, i.e. any open set $U \subseteq \Omega$ has $f(U)$ as an open set of $\mathbb{C}$.

Proof. Let $U \subseteq \Omega$ be an open set. We want to show that $f(U)$ is also an open set in $\mathbb{C}$, where f is holomorphic and nonconstant.

That is, we need to show that every $w_0 \in f(U)$ is an interior point; in a small disk around w_0, every point has a preimage in U.

Suppose $w_0 = f(z_0)$ for some $z_0 \in U$. This means that $f(z) - w_0$ has z_0 as a zero.

Consider a w close to w_0; we want $f(z) - w$ to also have some zero in U.

To show that this is the case, we can compare the holomorphic functions $f(z) - w$ and $f(z) - w_0$. Notice that

$$f(z) - w = (f(z) - w_0) + (w_0 - w).$$

Let $z_0 \in U$ be a zero of $f(z) - w_0$ (i.e. $f(z_0) = w_0$). Because U is open, there exists some $\delta > 0$ such that $D_\delta(z_0) \subseteq U$.

f is also not a constant function, so $f(z) - w_0)$ is not constant zero, so z_0 is an isolated zero of the function $f(z) - w_0$.

We can make $\delta > 0$ even smaller so that on $\overline{D_\delta(z_0)}$, z_0 is the unique zero of $f(z) - w_0$.

Suppose we define $\varepsilon > 0$ such that

$$\varepsilon = \frac{1}{2} \inf_{z \in \partial D_\delta(z_0)} \left| f(z) - w_0 \right|.$$

We then have for any $w \in D_\varepsilon(w_0)$ and $z \in \partial D_\delta(z_0)$ that

$$\left| f(z) - w_0 \right| \geq \inf_{z \in \partial D_\delta(z_0)} \left| f(z) - w_0 \right| = 2\varepsilon > \varepsilon > |w - w_0| = |w_0 - w|.$$

Applying Rouche's theorem on $D_\delta(z_0)$, we can say that $(f(z) - w_0) + (w_0 - w) = f(z) - w$ has the same number of zeros as $f(z) - w_0$.

Since $f(z) - w_0$ has a zero z_0, we know that $f(z) = w$ also has a solution in $D_\delta(z_0) \subseteq U$. This means that $w \in f(U)$, and thus $D_\varepsilon(w_0) \subseteq f(U)$, since $w \in D_\varepsilon(w_0)$ is arbitrary. This shows that w_0 is an interior point. □

As a remark, the preimage of an open set is always open for a continuous function. However, the image of an open set by a continuous function may not be open.

For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ has domain $\mathbb{R}$, which is open, but image $[0, \infty)$, which is not an open set.

Theorem 24.2: Maximum Modulus Principle

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with f holomorphic on Ω.

Then, $\left| f \right|$ has no maximum in Ω unless f is a constant function.

Proof. If f is a constant function, then we're done. If f is not a constant function, suppose for contradiction that $\left| f \right|$ attains its maximum at $z_0 \in \Omega$.

Suppose we take $D_r(z_0) \in \Omega$ for some $r > 0$. By the open mapping theorem, $f(D_r(z_0))$ is an open set in $\mathbb{C}$.

Specifically $f(z_0)$ is an interior point to $f(D_r(z_0))$. This means that we can find some $w \in f(D_r(z_0))$ with $|w| > \left| f(z_0) \right|$ (specifically, there must be some other point in $f(D_r(z_0))$ that is further away from the origin; otherwise, we're at a boundary and the set is not open).

However, this is impossible, and contradicts with the fact that $\left| f(z_0) \right|$ is the maximum of $\left| f \right|$. □

Example 24.3

For example, consider the function $f(z) = z^2 + 1$, with $\Omega = D_1(0)$.

$\left| f(z) \right| = \left| z^2 + 1 \right|$ has no maximum on $D_1(0)$; the maximum of $\left| f(z) \right|$ is achieved on $\partial D_1(0)$.

Corollary 24.4

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with $\overline{\Omega}$ a compact set (i.e. it is bounded).

Any holomorphic function f over Ωj that can be continuously extended to $\overline{\Omega}$ must achieve its maximum on $\partial\Omega$.

Proof. After a continuous extension to $\overline{\Omega}$, $\left| f \right|$ is a continuous function on $\overline{\Omega}$.

Since $\overline{\Omega}$ is compact, $\left| f \right|$ must attain its maximum at some point in $\overline{\Omega}$.

If the maximum is at $z_0 \in \Omega$, the maximum modulus principle says that f must be constant. This means that the maximum is also achieved at $\partial\Omega$.

Otherwise, the maximum occurs on $\partial\Omega$, and we are also done. □

24.1 Homotopy of paths

Definition 24.5: Homotopy

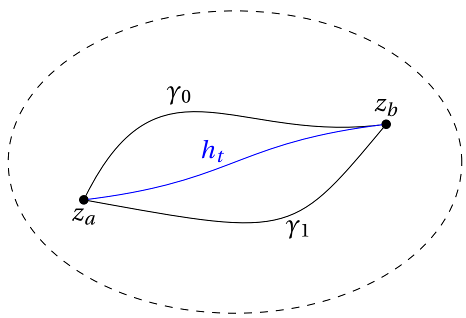
Suppose $\Omega \subseteq \mathbb{C}$ is an open set.

Let $\gamma_0, \gamma_1 : [a,b] \to \Omega$ be continuous paths, with $\gamma_0(a) = \gamma_1(a) = z_a$ and $\gamma_0(b) = \gamma_1(b) = z_b$.

We say that γ_0 and γ_1 are *homotopic* in Ω (relative to the endpoints) if there is a continuous map $h : [a,b] \times [0,1] \to \Omega$ such that $h(\cdot, 0) = \gamma_0$ and $h(\cdot, 1) = \gamma_1$.

Relative to the endpoints, this also means that $h(a,t) = z_a$ and $h(b,t) = z_b$ for all $t \in [0,1]$.

Visually, we have



Homotopy also gives us an *equivalence relation* for continuous paths in Ω. Specifically, the following properties hold:

- $\gamma \sim \gamma$

  With $h(s,t) = \gamma(s)$, a path is homotopic to itself.

- $\gamma_0 \sim \gamma_1 \implies \gamma_1 \sim \gamma_0$

  Suppose γ_0 is homotopic to γ_1, through $h : [a,b] \times [0,1] \to \Omega$.

  We can define $\overline{h} : [a,b] \times [0,1] \to \Omega$ such that $\overline{h}(\cdot, t) = h(\cdot, 1-t)$, essentially going backwards from γ_1 to γ_0.

- $\gamma_0 \sim \gamma_1 \land \gamma_1 \sim \gamma_2 \implies \gamma_0 \sim \gamma_2$

  Suppose h_1 goes from γ_0 to γ_1, and h_2 goes from γ_1 to γ_2. We can define

$$h(s,t) = \begin{cases} h_1(s, 2t) & t \in [0, \frac{1}{2}] \\ h_2(s, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

  which goes from γ_0 to γ_2.

Theorem 24.6: Homotopic invariance of contour integrals

For any two homotopic piecewisely smooth curves γ_0 and γ_1 which share the same endpoints,

$$\int_{\gamma_0} f(z_1)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z.$$

Proof. Suppose $h\colon [a,b]\times[0,1]\to\Omega$ is a homotopy for γ_0, γ_1.

Consider the map $\Phi\colon [0,1]\to\mathbb{C}$ defined as

$$\Phi(t) = \int_{h_t} f(z)\,\mathrm{d}z.$$

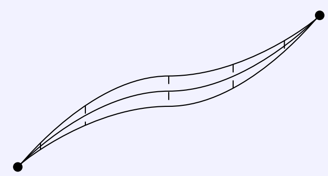
We want to show that Φ is a constant map.

Let us define $A := \Phi(0) = \int_{\gamma_0} f(z)\,\mathrm{d}z$. Consider the set $K = \{t\in[0,1] \mid \Phi(t) = A\}$.

We know that $K \neq \varnothing$, since $0 \in K$. We also know that $K \subseteq [0,1]$; if we can show that K is both open and closed, then we can conclude that $K = [0,1]$.

We can show that K is closed, since $K = \Phi^{-1}(\{A\})$, and $\{A\}$ is a closed set in $\mathbb{C}$.

To show that K is open, let us consider some $t \in K$. Drawing the path h_t and small deformations of the path, we have



Dividing up the pair of paths, we have a series of closed loops which integrate to zero. This means that together, the two deformations have the same contour integral.

Since the contour integral of the original line was A, it must then be the case that any small deformation preserves the same integral value, and as such t is interior to K. Since the choice of t was arbitrary, K must then be an open set.

We've shown that K is both open and closed, and the only possible K is $[0,1]$. □

4/21/2022

Lecture 25

Connectedness, Complex Logarithm

Definition 25.1: Simply Connected

A subset $\Omega\subseteq\mathbb{C}$ is *simply connected* if any two points with the same endpoints are homotopic to each other.

As an example, $\Omega = \mathbb{C}$ is simply connected.

Definition 25.2: Convex set

A subset $\Omega \subseteq \mathbb{C}$ is *convex* if for any two points $z_0, z_1 \in \Omega$, the line segment $(1-t)z_0 + tz_1 \in \Omega$ for any $t \in [0,1]$.

Lemma 25.3

Any convex set is simply connected.

Proof. Suppose $\Omega \subseteq \mathbb{C}$ is convex. Further, take any two continuous curves γ_0 and γ_1 with the same endpoints:

$$\begin{aligned}\gamma_0 &: [a,b] \to \Omega\\ \gamma_1 &: [a,b] \to \Omega\\ \gamma_0(a) &= \gamma_1(a)\\ \gamma_0(b) &= \gamma_1(b)\end{aligned}$$

We can construct a homotopy $h : [a,b] \times [0,1] \to \Omega$ such that

$$h(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s).$$

Here, for any $s \in [a,b]$, $\gamma_0(s), \gamma_1(s) \in \Omega$, and since Ω is convex, we also have that $(1-t)\gamma_0(s) + t\gamma_1(s) \in \Omega$ for all $t \in [0,1]$. □

Corollary 25.4

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, with f holomorphic on Ω.

If Ω is simply connected, then over any loop γ in Ω,

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. Suppose $\gamma(0) = z_0$. We know that γ is homotopic to a constant z_0 loop (i.e. the loop containing only z_0), so by a previous theorem, their integrals are equal:

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_{\{z_0\}} f(z)\,\mathrm{d}z = 0.$$

□

25.1 Complex Logarithm

The complex logarithm is the inverse of the exponential function

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Lemma 25.5

$f(z) = e^z$ is the unique function satisfying the following properties:

- $f : \mathbb{C} \to \mathbb{C}$, $f(\mathbb{R}) \subseteq \mathbb{R}$, $f(1) = e$
- $f(z_1 + z_2) = f(z_1)f(z_2)$

- f is an entire function

The question becomes: does $f(z) = e^z : \mathbb{C} \to \mathbb{C}$ have an inverse map?

That is, for any $w \in \mathbb{C}$, does $e^z = w$ have a unique solution?

We can write $w = re^{i\theta}$, and $z = x + iy$. We'd need

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = re^{i\theta}.$$

This means that $e^x = r$, and $e^{iy} = e^{i\theta}$. Solving for x and y, we have

$$\begin{aligned} x &= \ln r = \ln|w| \\ y &= \theta + 2k\pi && (k \in \mathbb{Z}) \end{aligned}$$

Here, ln is the logarithm in $\mathbb{R}$, which places a restriction that $w \neq 0$; this means that its image is $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$.

Since y is not unique either, e^z is neither injective nor surjective.

This means that for $w \in \mathbb{C}^*$, we define a multi-valued function

$$\operatorname{Log}(w) = \{z \in \mathbb{C} \mid e^z = w\} = \ln|w| + i \operatorname{Arg}(w),$$

where here $\operatorname{Arg}(w)$ is the multivalued function

$$\operatorname{Arg}(w) = \{\theta + 2k\pi \mid z \in \mathbb{Z}\},$$

for θ in $w = re^{i\theta}$, i.e. the counterclockwise angle from the positive real axis.

To make the functions well-defined, we also have the notion of the *principal logarithm* and the *principal argument*.

The principal logarithm is defined as

$$\log(w) = \ln|w| + i \arg(w),$$

where here $\arg(w)$ is the principal argument, i.e. Arg restricted to $[-\pi, \pi)$. (This differs by convention; some people use $[0, 2\pi)$.)

Example 25.6

Compute the Log and principal logs of the following:

1. $\log 1$, $\log e$, $\log 2$, $\operatorname{Log} 1$, $\operatorname{Log} e$, $\operatorname{Log} 2$

   We have

$$\begin{aligned} \log 1 &= \ln 1 + i \arg(1) \\ &= \ln 1 + i \cdot 0 = \boxed{0} \\ \operatorname{Log} 1 &= \{0 + i \cdot 2k\pi \mid k \in \mathbb{Z}\} \\ \log e &= \ln e + i \arg(e) = \boxed{1} \\ \operatorname{Log} e &= \{1 + i \cdot 2k\pi \mid k \in \mathbb{Z}\} \\ \log 2 &= \ln 2 + i \arg 2 = \boxed{\ln 2} \approx 0.69315 \\ \operatorname{Log} 2 &= \{\ln 2 + i \cdot 2k\pi \mid k \in \mathbb{Z}\} \end{aligned}$$

2. $\log(-1)$, $\log(-e)$, $\log(-2)$, $\operatorname{Log}(-1)$, $\operatorname{Log}(-e)$, $\operatorname{Log}(-2)$

We have

$$\begin{aligned}\log(-1) &= \ln 1 + i\arg(-1)\\ &= i(-\pi) = -\pi i\\ \operatorname{Log}(-1) &= \{i(-\pi+2k\pi) \mid k\in\mathbb{Z}\}\\ \log(-e) &= \ln e + i\arg(-e)\\ &= 1-\pi i\\ \operatorname{Log}(-e) &= \{1+i(-\pi+2k\pi) \mid k\in\mathbb{Z}\}\\ \log(-2) &= \ln 2 - \pi i\\ \operatorname{Log}(-2) &= \{\ln 2 + i(-\pi+2k\pi) \mid k\in\mathbb{Z}\}\end{aligned}$$

3. $\log(1+i)$, $\operatorname{Log}(1+i)$

   We have

$$\begin{aligned}\log(1+i) &= \ln|1+i| + i\arg(1+i)\\ &= \ln\sqrt{2}+\frac{\pi}{4}i\\ \operatorname{Log}(1+i) &= \{\ln\sqrt{2}+i(\frac{\pi}{4}+2k\pi) \mid k\in\mathbb{Z}\}\end{aligned}$$

Log also satisfies the following properties:

- $\operatorname{Log}(z_1z_2) = \operatorname{Log}(z_1)+\operatorname{Log}(z_2)$
- $\operatorname{Log}(\frac{z_1}{z_2}) = \operatorname{Log}(z_1)-\operatorname{Log}(z_2)$

Here, we define

$$\operatorname{Log}(z_1)+\operatorname{Log}(z_2) = \{w_1+w_2 \mid w\in\operatorname{Log}(z_1), w_2\in\operatorname{Log}(z_2)\}.$$

As a followup, here is an incorrect proof that $\operatorname{Log}(z) = \operatorname{Log}(-z)$:

"Proof".

$$\begin{aligned}\operatorname{Log}((-z)^2) &= \operatorname{Log}(z^2)\\ \operatorname{Log}((-z)(-z)) &= \operatorname{Log}(z\cdot z)\\ \operatorname{Log}(-z)+\operatorname{Log}(-z) &= \operatorname{Log}(z)+\operatorname{Log}(z)\\ 2\operatorname{Log}(-z) &= 2\operatorname{Log}(z)\\ \operatorname{Log}(-z) &= \operatorname{Log}(z)\end{aligned}$$

□

The error is the simplification in the second to last line. We cannot say that $\operatorname{Log}(z)+\operatorname{Log}(z) = 2\operatorname{Log}(z)$, by the definition earlier; the addition applies for every combination of elements in the set.

4/26/2022

Lecture 26

Branches of the Logarithm

Definition 26.1: Branch of the Logarithm

A continuous right inverse of the exponential function is called a *branch of the logarithm*.

Breaking down the above definition, a *right inverse* of the exponential function is a function "log" such that exp∘"log″ gives the identity function. We call this "log" function a *branch of the logarithm*.

More intuition for this definition can be gotten through the following theorem.

Theorem 26.2

Suppose $\Omega \subseteq \mathbb{C}$ is an open set, which is also connected and simply connected. Further, let $1 \in \Omega$ and $0 \notin \Omega$.

In Ω, there is a branch of the logarithm $F(z) = \log_\Omega(z)$ such that

1. F is holomorphic on Ω
2. F is a right inverse of exp, i.e. $e^{F(z)} = z$ for all $z \in \Omega$
3. $F(r) = \ln r$ for $r \in \mathbb{R} \cap \Omega$ and near 1 (that is, $F(1) = \ln 1 = 0$)

Moreover, such an F is the *unique* function that satisfies these properties.

Proof. We will prove the existence of such an F function by constructing F explicitly.

Suppose we take any $z \in \Omega$; since Ω is open and connected, it is path-connected, so we can take a piecewisely smooth path connecting 1 and z. Let us name this path $\gamma(z)$.

We then define

$$F(z) = \int_{\gamma(z)} \frac{1}{w}\,\mathrm{d}w.$$

Firstly, we claim that $F(z)$ is independent of our choice of path.

Since Ω is simply connected, any two paths connecting 1 and z in Ω are homotopic to each other. Further, we have that $\frac{1}{w}$ is holomorphic on Ω, since $0 \notin \Omega$. This shows that $\int_{\gamma(z)} \frac{1}{w}\,\mathrm{d}w$ has the same value, regardless of our choice of path.

Because of this, we have a map $F : \Omega \to \mathbb{C}$.

We can also show that F is holomorphic on Ω, following a similar proof as with Goursat's theorem. This means that $F'(z) = \frac{1}{z}$.

Next, we'll show that $e^{F(z)} = z$ for all $z \in \Omega$ (i.e. that F is a right inverse of exp).

Suppose we consider $g(z) := e^{-F(z)} \cdot z$. Then,

$$\begin{aligned} g'(z) &= e^{-F(z)}(-F'(z)) \cdot z + e^{-F(z)} \\ &= e^{-F(z)}\left(-F'(z) \cdot z + 1\right) \\ &= e^{-F(z)}\left(-\frac{1}{z} \cdot z + 1\right) = 0 \end{aligned}$$

This means that $g'(z) = 0$ for all $z \in \Omega$. Since Ω is connected, this means that g must be a constant function on Ω. The constant is then

$$g(1) = e^{-F(1)} = e^{-0} = 1.$$

This means that for any $z \in \Omega$, $g(z) = e^{-F(z)} \cdot z = 1 \implies e^{F(z)} = z$.

Next, we'll show that $F(r) = \ln r$ for $r \in \mathbb{R} \cap \Omega$ near 1.

Suppose we take any $r \in \mathbb{R}$ close to 1; since Ω is open, there will always be some open disk centered at 1 in Ω including r. This means that the line segment from 1 to r is in Ω.

Let us denote $\gamma(r)$ as this line segment. We can parameterize the line segment with

$$w(t) = (1-t)\cdot 1 + t \cdot r : [0,1] \to \mathbb{R}.$$

By our definition of $F(r)$, we have

$$\begin{aligned} F(r) &= \int_{\gamma(r)} \frac{1}{w}\,\mathrm{d}w \\ &= \int_0^1 \frac{r-1}{(1-t)\cdot 1 + t\cdot r}\,\mathrm{d}t \\ &= \int_0^1 \frac{r-1}{(r-1)t+1}\,\mathrm{d}t \\ &= \ln((r-1)t+1)\big|_{t=0}^{t=1} \\ &= \ln r \end{aligned}$$

To show that $F(z)$ is unique, let us consider two maps $F_1(z)$ and $F_2(z)$ which satisfy all of the listed properties.

We know that both functions are holomorphic on Ω, and both functions are equal to $\ln r$ for $r \in \mathbb{R}$ in Ω near 1. Applying the unique continuation theorem, we can see that $F_1(z) = F_2(z)$ on Ω. □

One thing to notice here is that $\log_\Omega(z)$ depends on the choice of Ω.

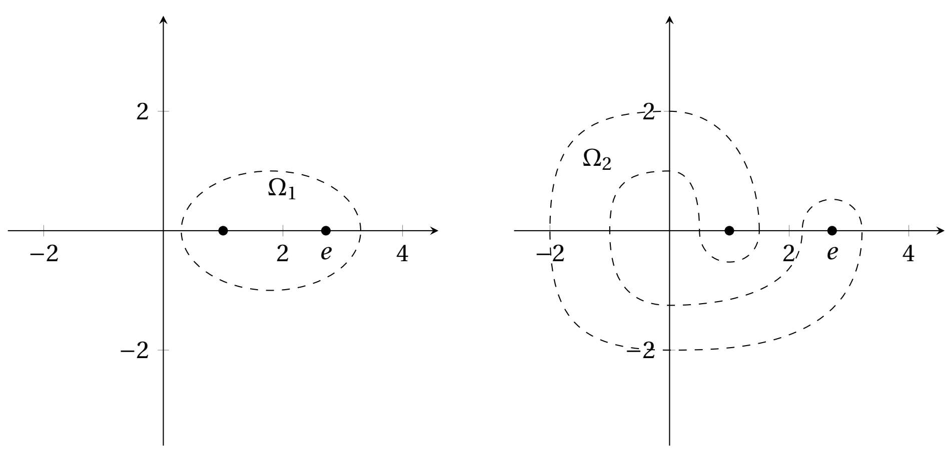


Figure 26.1: Comparison of two different branches of the logarithm

Figure 26.1 shows two different branches of the logarithm. Consider $\log_{\Omega_1}(e)$ and $\log_{\Omega_2}(e)$.

With the left graph, e is close to 1, and we can draw a line segment on the real axis connecting 1 and e. This means that $\log_{\Omega_1}(e) = \ln e = 1$.

However, with the right graph, e is not close to 1; we cannot draw the line segment on the real axis connecting 1 and e, so $\log_{\Omega_2}(e) \neq \ln e$.

It actually turns out that the value of log depends only on how many times we revolve around the pole $z = 0$ of $\frac{1}{z}$; each time we revolve around $z = 0$, we add $2\pi i$ to the value of $\log(z)$. As such, it turns out that $\log_{\Omega_2}(e) = 1 + 2\pi i$.

We'll now talk about a generalization of the previous theorem.

Theorem 26.3

Suppose $\Omega \subseteq \mathbb{C}$ is an open subset, and is connected and simply connected. Let $f(z)$ be a nowhere vanishing holomorphic function on Ω.

There exists a holomorphic function g on Ω such that

$$e^{g(z)} = f(z),$$

for all $z \in \Omega$.

Such a function g is not unique, but all such functions are shifts by multiples of $2\pi i$; the set of all such functions is

$$\{g(z) + 2\pi i \cdot k \mid k \in \mathbb{Z}\}.$$

Proof. Suppose we take any $z_0 \in \Omega$ with some $c_0 \in \mathbb{C}$ such that $e^{c_0} = f(z_0)$.

Let us define

$$g(z) = \int_{\gamma_{z_0}(z)} \frac{f'(w)}{f(w)} \, \mathrm{d}w + c_0.$$

Here, $\gamma_{z_0}(z)$ is a piecewise smooth path connecting z_0 and z.

Since Ω is simply connected, $g(z)$ is independent of the choice of such a path. Since f is also assumed to be nowhere vanishing and holomorphic, it must be the case that $g(z)$ is holomorphic, and $g'(z) = \frac{f'(z)}{f(z)}$.

We will now check that $e^{g(z)} = f(z)$ for all $z \in \Omega$.

Consider the function $h(z) = e^{-g(z)} f(z)$. We have

$$\begin{aligned} h'(z) &= -g'(z) e^{-g(z)} f(z) + e^{-g(z)} f'(z) \\ &= e^{-g(z)} \big(-g'(z) \cdot f(z) + f'(z)\big) \\ &= e^{-g(z)} \left(-\frac{f'(z)}{f(z)} + f'(z)\right) \\ &= 0 \end{aligned}$$

Since Ω is connected and we've just shown that $h'(z) = 0$, it must be the case that h is a constant function. This means that its value is equal to

$$h(z) = h(z_0) = e^{-g(z_0)} \cdot f(z_0) = e^{-c_0} \cdot e^{c_0} = 1.$$

This shows that $h(z) = 1$ on Ω, and thus $e^{g(z)} = f(z)$ on Ω.

We can further see that if $g(z)$ is a solution, then $g(z) + 2\pi i \cdot k$ for $k \in \mathbb{Z}$ are also solutions:

$$e^{g(z) + 2\pi i k} = e^{g(z)} \cdot e^{2\pi i k} = e^{g(z)} = f(z).$$

Conversely, if $g_1(z)$ and $g_2(z)$ are both solutions, then

$$e^{g_1(z)} = f(z) = e^{g_2(z)} \implies e^{g_1(z) - g_2(z)} = 1.$$

This means that $g_1(z) - g_2(z) \in \{2\pi i \cdot k \mid k \in \mathbb{Z}\}$. □

As a closing remark, recall how we define z^α for $\alpha \in \mathbb{R}$; we can also extend this idea to $\alpha \in \mathbb{C}$ through the logarithm. We define $z^\alpha = e^{\alpha \cdot \operatorname{Log} z}$. Specifically, over a branch Ω, we define $z_\Omega^\alpha = e^{\alpha \log_\Omega(z)}$.

If we consider $\Omega = \mathbb{C} \setminus (-\infty, 0]$, it turns out that we have

$$\log_{\mathbb{C}\setminus(-\infty,0]}(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n},$$

with convergence radius 1. The shift allows for the function to be defined at a disk of radius 1 centered at 1.

4/28/2022

Lecture 27

Conformal Mappings

Definition 27.1: Conformal Mapping

Suppose $U, V \subseteq \mathbb{C}$ are open sets.

A function $f: U \to V$ is called a *conformal map* or a *biholomorphic map* if

1. f is bijective
2. f is holomorphic
3. f^{-1} is holomorphic

Here, note that the inverse map f^{-1} will always exist because f is bijective.

Definition 27.2: Conformal Equivalence

Two sets U and V are *conformally equivalent* if there is a conformal map from U to V.

Lemma 27.3

Conformal equivalence is an equivalence relation;

- $\mathrm{id}_U: U \to U$
- $f: U \to V$ is conformal means that $f^{-1}: V \to U$ is conformal
- $f: U \to V$ and $g: V \to W$ are conformal means that $g \circ f: U \to W$ is also conformal

Complex analysis mainly studies conformal equivalence classes.

Lemma 27.4

Suppose $U, V \subseteq \mathbb{C}$ are open sets, with $f: U \to V$ injective and holomorphic. Then, f' is nowhere vanishing on U.

Proof. We will prove this through contradiction. If the claim is false, then there must be some point $z_0 \in U$ such that $f'(z_0) = 0$.

The injectivity of f implies that z_0 is an isolated zero for the holomorphic function f' (otherwise, $f' = 0$ in a neighborhood of z_0, and f is not injective)).

This implies that there is a small disk $D(z_0) \subseteq U$ such that z_0 is the only zero of f' on $D(z_0)$.

Since f is holomorphic (thus analytic) on $D(z_0)$, we can write

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$$

$$= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \cdots$$
$$= f(z_0) + a_k(z - z_0)^k + g(z)$$

The last equality comes from the fact that $f'(z_0) = 0$; here, we have $k \geq 2$ be the first instance in which $a_k \neq 0$. $g(z)$ encompasses all higher order terms.

Now, let us consider some small $\varepsilon > 0$ such that on $\partial D_\varepsilon(z_0)$,

$$\left|a_k(z - z_0)^k\right| > \left|g(z)\right|.$$

Intuitively, we know that such an ε exists if we consider

$$\frac{g(z)}{a_k(z - z_0)^k} \xrightarrow{z \to z_0} 0.$$

This is because the fraction turns into $(z - z_0)$ raised to some positive power (as $g(z)$ contains terms of higher order than k), multiplied by some other constants; for z close to z_0, the expression tends toward zero. This means that if we choose ε small enough, we will have values of $\frac{|g(z)|}{|a_k(z-z_0)^k|}$ less than 1 within the disk $D_\varepsilon(z_0)$.

Let us define

$$\delta := \inf_{z \in \partial D_\varepsilon(z_0)} \{\left|a_k(z - z_0)^k\right| - \left|g(z)\right|\} > 0.$$

Note that the elements of this set are all positive everywhere on $\partial D_\varepsilon(z_0)$.

We claim that for any w with $0 < |w| < \frac{1}{2}\delta$,

$$\left|g(z) - w\right| < \left|a_k(z - z_0)^k\right|,$$

for any $z \in \partial D_\varepsilon(z_0)$.

To see this, we have by the triangle inequality

$$\begin{aligned}
\left|g(z) - w\right| &\leq \left|g(z)\right| + |w| \\
&= \left|a_k(z - z_0)^k\right| + |w| + \left(\left|g(z)\right| - \left|a_k(z - z_0)^k\right|\right) \\
&= \left|a_k(z - z_0)^k\right| + \underbrace{|w|}_{\leq \frac{1}{2}\delta} - \underbrace{\left(\left|a_k(z - z_0)^k\right| - \left|g(z)\right|\right)}_{\geq \delta} \\
&\leq \left|a_k(z - z_0)^k\right| - \frac{1}{2}\delta \\
&< \left|a_k(z - z_0)^k\right|
\end{aligned}$$

Now, consider the two functions $a_k(z - z_0)^k + g(z) - w$ and $a_k(z - z_0)^k$. The former is equal to

$$\underbrace{a_k(z - z_0)^k + g(w)}_{f(z) - f(z_0)} - w = f(z) - f(z_0) - w,$$

by the power series expansion from earlier. Further, from our most recent result, we know that $\left|a_k(z - z_0)^k\right| > \left|g(z) - w\right|$, so by Rouche's theorem, these two functions have the same number of zeroes in $D_\varepsilon(z_0)$.

Further, $a_k(z - z_0)^k = 0$ has k equal zeroes as $z = z_0$ in $D_\varepsilon(z_0)$, which means that $f(z) - f(z_0) - w = 0$ has k zeros as well.

Now, we claim that $f(z) - f(z_0) - w$ has no multiple zeroes in $D_\varepsilon(z_0)$; that is, there does not exist any zero of order ≥ 2.

To see this, suppose for contradiction that $h(z) = f(z) - f(z_0) - w$ has a multiple zero at $z' \neq z_0$, for $z' \in D_\varepsilon(z_0)$. Specifically, we have $h(z') = 0$ of order $n \geq 2$.

Near z', we can express $h(z) = (z - z')^n \phi(z)$, where n is the order of the zero, and $\phi(z)$ is nowhere vanishing near z'. Taking the derivative, we have

$$h'(z) = n(z - z')^{n-1}\phi(z) + (z - z')^n \phi'(z).$$

This means that at z', we have $h'(z') = 0$.

However, we also have $h'(z) = \big(f(z) - f(z_0) - w\big)' = f'(z)$. This means that we've just shown that $f'(z') = 0$. However, we've asserted earlier that z_0 is the only zero of f' on $D(z_0)$. This is a contradiction, and as such $f(z) - f(z_0) - w$ cannot have any zero of order ≥ 2.

Since we've shown that $f(z) - f(z_0) - w$ must have k zeroes, while also does not have any zero of order ≥ 2, it must be the case that there are $z_1 \neq z_2$ such that both z_1 and z_2 are zeroes of $f(z) - f(z_0) - w$. This means that we have

$$f(z_1) - f(z_0) - w = 0 = f(z_2) - f(z_0) - w \implies f(z_1) = f(z_2).$$

This contradicts with the assumption that f is injective on U, and as such the original claim holds. □

Corollary 27.5

Suppose $U, V \subseteq \mathbb{C}$ are open sets, and $f : U \to V$ is injective and holomorphic.

Then, f is conformal (biholomorphic) onto its image.

Proof. Since f is holomorphic, and U is open, we can see that $f(U)$ is also open by the open mapping theorem. This means that $f : U \to f(U) \subseteq V$ is a bijective map (f is injective by assumption, and f is always surjective onto its image).

This means that we only need to check that $g = f^{-1} : f(U) \to U$ is holomorphic.

Suppose we take any $w_0 \in f(U)$, and let $w_0 = f(z_0)$ for some $z_0 \in U$.

We have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{g(w) - g(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Here, the last equality is due to the fact that we've define $g = f^{-1}$, meaning $g(w) = f^{-1}(w) = z$ and $f(z) = w$.

As $w \to w_0$, we have that $z \to z_0$, and $\frac{f(z) - f(z_0)}{z - z_0} \to f'(z_0)$. From the previous lemma, we also know that $f'(z_0) \neq 0$.

This means that

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{f'(z_0)}.$$

This shows that g is differentiable at any arbitrary $w_0 \in f(U)$, and thus is holomorphic. □